CHAPTER 17

Markov Processes

CONTENTS

Markov process models are useful in studying the evolution of systems over repeated trials. The repeated trials are often successive time periods in which the state of the system in any particular period cannot be determined with certainty. Rather, transition probabilities are used to describe the manner in which the system makes transitions from one period to the next. Hence, we are interested in the probability of the system being in a particular state at a given time period.

Markov process models can be used to describe the probability that a machine that is functioning in one period will continue to function or will break down in the next period. Models can also be used to describe the probability that a consumer purchasing brand A in one period will purchase brand B in the next period. The Q.M. in Action, Benefit of Health Care Services, describes how a Markov process model was used to determine the health status probabilities for persons age 65 and older. Such information was helpful in understanding the future need for health care services and the benefits of expanding current health care programs.

In this chapter we present a marketing application that involves an analysis of the store-switching behavior of supermarket customers. As a second illustration, we consider an accounting application that is concerned with the transitioning of accounts receivable dollars to different account-aging categories. Because an in-depth treatment of Markov processes is beyond the scope of this text, the analysis in both illustrations is restricted to situations consisting of a finite number of states, the transition probabilities remaining constant over time, and the probability of being in a particular state at any one time period depending only on the state in the immediately preceding time period. Such Markov processes are referred to as **Markov chains with stationary transition probabilities**.

Q.M. *in* ACTION

*BENEFIT OF HEALTH CARE SERVICES**

The U.S. Government Accountability Office (GAO) is an independent, nonpolitical audit organization in the legislative branch of the federal government. GAO evaluators obtained data on the health conditions of individuals age 65 and older. The individuals were identified as being in three possible states:

Best: Able to perform daily activities without assistance

Next Best: Able to perform some daily activities without assistance

Worst: Unable to perform daily activities without assistance

Using a two-year period, the evaluators developed estimates of the transition probabilities among the three states. For example, a transition probability that a person in the Best state is still in the Best state one year later was 0.80, while the transition probability that a person in the Best state moves to the Next Best state one year later is 0.10. The Markov analysis of the full set of transition probabilities determined the steady-state probabilities that individuals would be in each state. Thus, for a given population age 65 and older, the steady-state probabilities would indicate the percentage of the population that would be in each state in future years.

The GAO study further subdivided individuals into two groups: those receiving appropriate health care and those not receiving appropriate health care. For individuals not receiving appropriate health care, the kind of additional care and the cost of that care were estimated. The revised transition probabilities showed that with appropriate health care, the steady-state probabilities indicated the larger percentage of the population that would be in the Best and Next Best health states in future years. Using these results, the model provided evidence of the future benefits that would be achieved by expanding current health care programs.

*Based on information provided by Bill Ammann, U.S. Government Accountability Office.

17.1 Market Share Analysis

Suppose we are interested in analyzing the market share and customer loyalty for Murphy's Foodliner and Ashley's Supermarket, the only two grocery stores in a small town. We focus on the sequence of shopping trips of one customer and assume that the customer makes one shopping trip each week to either Murphy's Foodliner or Ashley's Supermarket, but not both.

Using the terminology of Markov processes, we refer to the weekly periods or shopping trips as the **trials of the process**. Thus, at each trial, the customer will shop at either Murphy's Foodliner or Ashley's Supermarket. The particular store selected in a given week is referred to as the **state of the system** in that period. Because the customer has two shopping alternatives at each trial, we say the system has two states. With a finite number of states, we identify the states as follows:

State 1. The customer shops at Murphy's Foodliner
State 2. The customer shops at Ashley's Supermarket

If we say the system is in state 1 at trial 3, we are simply saying that the customer shops at Murphy's during the third weekly shopping period.

As we continue the shopping trip process into the future, we cannot say for certain where the customer will shop during a given week or trial. In fact, we realize that during any given week, the customer may be either a Murphy's customer or an Ashley's customer. However, using a Markov process model, we will be able to compute the probability that the customer shops at each store during any period. For example, we may find a 0.6 probability that the customer will shop at Murphy's during a particular week and a 0.4 probability that the customer will shop at Ashley's.

To determine the probabilities of the various states occurring at successive trials of the Markov process, we need information on the probability that a customer remains with the same store or switches to the competing store as the process continues from trial to trial or week to week.

Suppose that, as part of a market research study, we collect data from 100 shoppers over a 10-week period. Suppose further that these data show each customer's weekly shopping trip pattern in terms of the sequence of visits to Murphy's and Ashley's. To develop a Markov process model for the sequence of weekly shopping trips, we need to express the probability of selecting each store (state) in a given period solely in terms of the store (state) that was selected during the previous period. In reviewing the data, suppose that we find that of all customers who shopped at Murphy's in a given week, 90% shopped at Murphy's the following week while 10% switched to Ashley's. Suppose that similar data for the customers who shopped at Ashley's in a given week show that 80% shopped at Ashley's the following week while 20% switched to Murphy's. Probabilities based on these data are shown in Table 17.1. Because these probabilities indicate that a customer moves, or makes a transition,

TABLE 17.1 TRANSITION PROBABILITIES FOR MURPHY'S AND ASHLEY'S GROCERY SALES

Current Weekly Shopping Period	Next Weekly Shopping Period: Murphy's Foodliner	Next Weekly Shopping Period: Ashley's Supermarket
Murphy's Foodliner	0.9	0.1
Ashley's Supermarket	0.2	0.8

from a state in a given period to each state in the following period, these probabilities are called **transition probabilities**.

An important property of the table of transition probabilities is that the sum of the probabilities in each row is 1; each row of the table provides a conditional probability distribution. For example, a customer who shops at Murphy's one week must shop at either Murphy's or Ashley's the next week. The entries in row 1 give the probabilities associated with each of these events. The 0.9 and 0.8 probabilities in Table 17.1 can be interpreted as measures of store loyalty in that they indicate the probability of a repeat visit to the same store in consecutive weeks. Similarly, the 0.1 and 0.2 probabilities are measures of the store-switching characteristics of the customers. In developing a Markov process model for this problem, we are assuming that the transition probabilities will be the same for any customer and that the transition probabilities will not change over time.

Appendix 17.1 contains a review of matrix notation and operations.

Note that Table 17.1 has one row and one column for each state of the system. We will use the symbol p_{ij} to represent the transition probabilities and the symbol P to represent the matrix of transition probabilities; that is,

$$p_{ij} = \text{probability of making a transition from state } i \text{ in a given period to state } j \text{ in the next period}$$

For the supermarket problem, we have

A quick check for a valid matrix of transition probabilities is to make sure the sum of the probabilities in each row equals 1.

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

Using the matrix of transition probabilities, we can now determine the probability that a customer will be a Murphy's customer or an Ashley's customer at some period in the future. Let us begin by assuming that we have a customer whose last weekly shopping trip was to Murphy's. What is the probability that this customer will shop at Murphy's on the next weekly shopping trip, period 1? In other words, what is the probability that the system will be in state 1 after the first transition? The matrix of transition probabilities indicates that this probability is $p_{11} = 0.9$.

Now let us consider the state of the system in period 2. A useful way of depicting what can happen on the second weekly shopping trip is to draw a tree diagram of the possible outcomes (see Figure 17.1). Using this tree diagram, we see that the probability that the customer shops at Murphy's during both the first and the second weeks is $(0.9)(0.9) = 0.81$. Also, note that the probability of the customer switching to Ashley's on the first trip and then switching back to Murphy's on the second trip is $(0.1)(0.2) = 0.02$. Because these options are the only two ways that the customer can be in state 1 (shopping at Murphy's) during the second period, the probability of the system being in state 1 during the second period is $0.81 + 0.02 = 0.83$. Similarly, the probability of the system being in state 2 during the second period is $(0.9)(0.1) + (0.1)(0.8) = 0.09 + 0.08 = 0.17$.

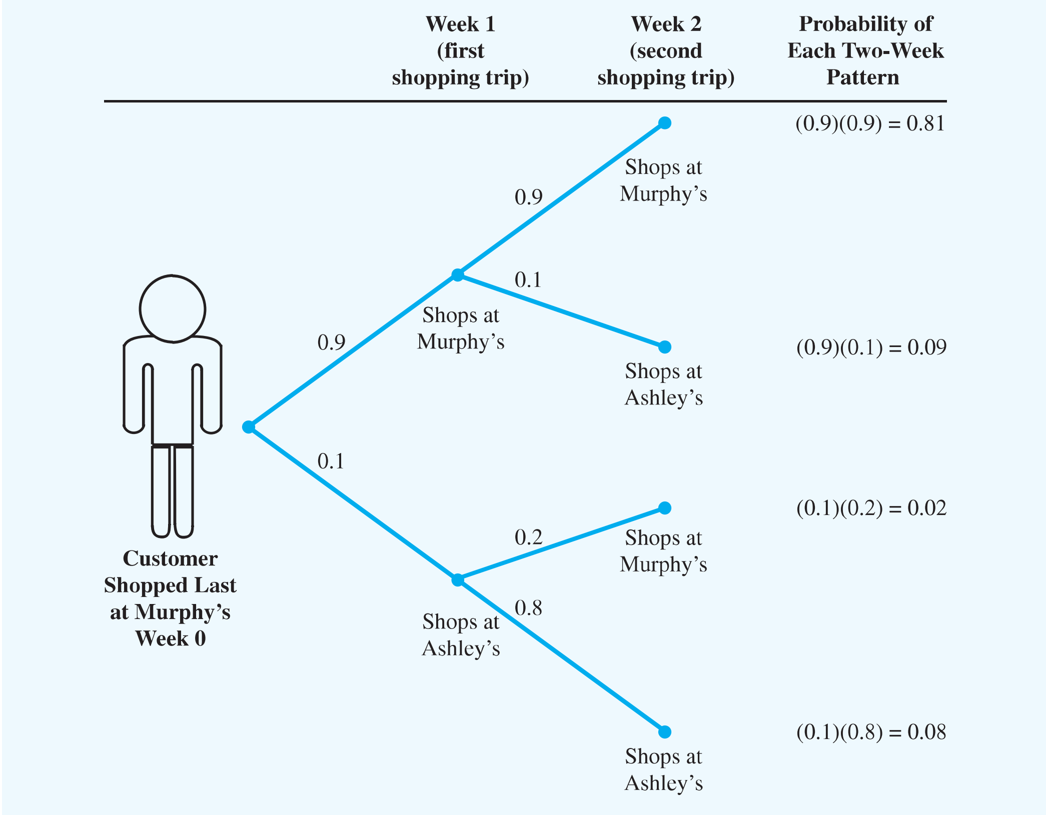
As desirable as the tree diagram approach may be from an intuitive point of view, it becomes cumbersome when we want to extend the analysis to three or more periods. Fortunately, we have an easier way to calculate the probabilities of the system being in state 1 or state 2 for any subsequent period. First, we introduce a notation that will allow us to represent these probabilities for any given period. Let

$$\pi_i(n) = \text{probability that the system is in state } i \text{ in period } n$$

Index denotes the state

Denotes the time period or number of transitions

FIGURE 17.1 TREE DIAGRAM DEPICTING TWO WEEKLY SHOPPING TRIPS OF A CUSTOMER WHO SHOPPED LAST AT MURPHY'S



For example, $\pi_1(1)$ denotes the probability of the system being in state 1 in period 1, while $\pi_2(1)$ denotes the probability of the system being in state 2 in period 1. Because $\pi_i(n)$ is the probability that the system is in state i in period n, this probability is referred to as a **state probability**.

The terms $\pi_1(0)$ and $\pi_2(0)$ will denote the probability of the system being in state 1 or state 2 at some initial or starting period. Week 0 represents the most recent period, when we are beginning the analysis of a Markov process. If we set $\pi_1(0) = 1$ and $\pi_2(0) = 0$, we are saying that as an initial condition the customer shopped last week at Murphy's; alternatively, if we set $\pi_1(0) = 0$ and $\pi_2(0) = 1$, we would be starting the system with a customer who shopped last week at Ashley's. In the tree diagram of Figure 17.1, we consider the situation in which the customer shopped last at Murphy's. Thus,

$$[\pi_1(0) \quad \pi_2(0)] = [1 \quad 0]$$

is a vector that represents the initial state probabilities of the system. In general, we use the notation

$$\Pi(n) = [\pi_1(n) \quad \pi_2(n)]$$

to denote the vector of state probabilities for the system in period n. In the example, $\Pi(1)$ is a vector representing the state probabilities for the first week, $\Pi(2)$ is a vector representing the state probabilities for the second week, and so on.

Using this notation, we can find the state probabilities for period $n + 1$ by simply multiplying the known state probabilities for period n by the transition probability matrix.

Using the vector of state probabilities and the matrix of transition probabilities, the multiplication[1] can be expressed as follows:

$$\Pi(\text{next period}) = \Pi(\text{current period})P$$

or

$$\Pi(n + 1) = \Pi(n)P \qquad \textbf{(17.1)}$$

Beginning with the system in state 1 at period 0, we have $\Pi(0) = [1 \ 0]$. We can compute the state probabilities for period 1 as follows:

$$\Pi(1) = \Pi(0)P$$

or

$$\begin{aligned} [\pi_1(1) \quad \pi_2(1)] &= [\pi_1(0) \quad \pi_2(0)]\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \\ &= [1 \quad 0]\begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \\ &= [0.9 \quad 0.1] \end{aligned}$$

The state probabilities $\pi_1(1) = 0.9$ and $\pi_2(1) = 0.1$ are the probabilities that a customer who shopped at Murphy's during week 0 will shop at Murphy's or at Ashley's during week 1.

Using equation (17.1), we can compute the state probabilities for the second week as follows:

$$\Pi(2) = \Pi(1)P$$

or

$$\begin{aligned} [\pi_1(2) \quad \pi_2(2)] &= [\pi_1(1) \quad \pi_2(1)]\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \\ &= [0.9 \quad 0.1]\begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \\ &= [0.83 \quad 0.17] \end{aligned}$$

We see that the probability of shopping at Murphy's during the second week is 0.83, while the probability of shopping at Ashley's during the second week is 0.17. These same results were previously obtained using the tree diagram of Figure 17.1. By continuing to apply equation (17.1), we can compute the state probabilities for any future period; that is,

$$\begin{aligned} \Pi(3) &= \Pi(2)P \\ \Pi(4) &= \Pi(3)P \\ \vdots \quad & \quad \vdots \\ \Pi(n + 1) &= \Pi(n)P \end{aligned}$$

Table 17.2 shows the result of carrying out these calculations for 10 periods.

[1]Appendix 17.1 provides the step-by-step procedure for vector and matrix multiplication.

TABLE 17.2 STATE PROBABILITIES FOR FUTURE PERIODS BEGINNING INITIALLY WITH A MURPHY'S CUSTOMER

State Probability	Period (n) 0	1	2	3	4	5	6	7	8	9	10
$\pi_1(n)$	1	0.9	0.83	0.781	0.747	0.723	0.706	0.694	0.686	0.680	0.676
$\pi_2(n)$	0	0.1	0.17	0.219	0.253	0.277	0.294	0.306	0.314	0.320	0.324

The vectors $\Pi(1)$, $\Pi(2)$, $\Pi(3)$, . . . contain the probabilities that a customer who started out as a Murphy customer will be in state 1 or state 2 in the first period, the second period, the third period, and so on. In Table 17.2 we see that after a few periods these probabilities do not change much from one period to the next.

If we had started with 1000 Murphy customers—that is, 1000 customers who last shopped at Murphy's—our analysis indicates that during the fifth weekly shopping period, on average, 723 would be customers of Murphy's, and 277 would be customers of Ashley's. Moreover, during the tenth weekly shopping period, 676 would be customers of Murphy's, and 324 would be customers of Ashley's.

Now let us repeat the analysis, but this time we will begin the process with a customer who shopped last at Ashley's. Thus,

$$\Pi(0) = [\pi_1(0) \quad \pi_2(0)] = [0 \ \ 1]$$

Using equation (17.1), the probability of the system being in state 1 or state 2 in period 1 is given by

$$\Pi(1) = \Pi(0)P$$

or

$$\begin{aligned}[\pi_1(1) \quad \pi_2(1)] &= [\pi_1(0) \quad \pi_2(0)]\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \\ &= [0 \quad 1]\begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \\ &= [0.2 \quad 0.8]\end{aligned}$$

Proceeding as before, we can calculate subsequent state probabilities. Doing so, we obtain the results shown in Table 17.3.

In the fifth shopping period, the probability that the customer will be shopping at Murphy's is 0.555, and the probability that the customer will be shopping at Ashley's is

TABLE 17.3 STATE PROBABILITIES FOR FUTURE PERIODS BEGINNING INITIALLY WITH AN ASHLEY'S CUSTOMER

State Probability	Period (n) 0	1	2	3	4	5	6	7	8	9	10
$\pi_1(n)$	0	0.2	0.34	0.438	0.507	0.555	0.589	0.612	0.628	0.640	0.648
$\pi_2(n)$	1	0.8	0.66	0.562	0.493	0.445	0.411	0.388	0.372	0.360	0.352

0.445. In the tenth period, the probability that a customer will be shopping at Murphy's is 0.648, and the probability that a customer will be shopping at Ashley's is 0.352.

As we continue the Markov process, we find that the probability of the system being in a particular state after a large number of periods is independent of the beginning state of the system. The probabilities that we approach after a large number of transitions are referred to as the **steady-state probabilities**. We shall denote the steady-state probability for state 1 with the symbol π_1 and the steady-state probability for state 2 with the symbol π_2. In other words, in the steady-state case, we simply omit the period designation from $\pi_i(n)$ because it is no longer necessary.

Analyses of Tables 17.2 and 17.3 indicate that as n gets larger, the difference between the state probabilities for period n and period $(n + 1)$ becomes increasingly smaller. This analysis leads us to the conclusion that as n gets large, the state probabilities for period $(n + 1)$ are very close to those for period n. This observation provides the basis of a simple method for computing the steady-state probabilities without having to actually carry out a large number of calculations.

In general, we know from equation (17.1) that

$$[\pi_1(n+1) \quad \pi_2(n+1)] = [\pi_1(n) \quad \pi_2(n)]\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

Because for sufficiently large n the difference between $\Pi(n + 1)$ and $\Pi(n)$ is negligible, we see that in the steady state $\pi_1(n + 1) = \pi_1(n) = \pi_1$ and $\pi_2(n + 1) = \pi_2(n) = \pi_2$. Thus, we have

$$\begin{aligned} [\pi_1 \quad \pi_2] &= [\pi_1 \quad \pi_2]\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \\ &= [\pi_1 \quad \pi_2]\begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \end{aligned}$$

After carrying out the multiplications, we obtain

$$\pi_1 = 0.9\pi_1 + 0.2\pi_2 \qquad \textbf{(17.2)}$$

and

$$\pi_2 = 0.1\pi_1 + 0.8\pi_2 \qquad \textbf{(17.3)}$$

However, we also know the steady-state probabilities must sum to 1 so

$$\pi_1 + \pi_2 = 1 \qquad \textbf{(17.4)}$$

Using equation (17.4) to solve for π_2 and substituting the result in equation (17.2), we obtain

Can you now compute the steady-state probabilities for Markov processes with two states? Problem 3 provides an application.

$$\begin{aligned} \pi_1 &= 0.9\pi_1 + 0.2(1 - \pi_1) \\ \pi_1 &= 0.9\pi_1 + 0.2 - 0.2\pi_1 \\ \pi_1 - 0.7\pi_1 &= 0.2 \\ 0.3\pi_1 &= 0.2 \\ \pi_1 &= \tfrac{2}{3} \end{aligned}$$

Then, using equation (17.4), we can conclude that $\pi_2 = 1 - \pi_1 = 1/3$. Thus, using equations (17.2) and (17.4), we can solve for the steady-state probabilities directly. You can check for yourself that we could have obtained the same result using equations (17.3) and (17.4).[2]

Thus, if we have 1000 customers in the system, the Markov process model tells us that in the long run, with steady-state probabilities $\pi_1 = 2/3$ and $\pi_2 = 1/3$, $2/3(1000) = 667$ customers will be Murphy's and $1/3(1000) = 333$ customers will be Ashley's. The steady-state probabilities can be interpreted as the market shares for the two stores.

Market share information is often quite valuable in decision making. For example, suppose Ashley's Supermarket is contemplating an advertising campaign to attract more of Murphy's customers to its store. Let us suppose further that Ashley's believes this promotional strategy will increase the probability of a Murphy's customer switching to Ashley's from 0.10 to 0.15. The revised transition probabilities are given in Table 17.4.

With three states, the steady-state probabilities are found by solving three equations for the three unknown steady-state probabilities. Try Problem 8 as a slightly more difficult problem involving three states.

Given the new transition probabilities, we can modify equations (17.2) and (17.4) to solve for the new steady-state probabilities or market shares. Thus, we obtain

$$\pi_1 = 0.85\pi_1 + 0.20\pi_2$$

Substituting $\pi_2 = 1 - \pi_1$ from equation (17.4), we have

$$\begin{aligned} \pi_1 &= 0.85\pi_1 + 0.20(1 - \pi_1) \\ \pi_1 &= 0.85\pi_1 + 0.20 - 0.20\pi_1 \\ \pi_1 - 0.65\pi_1 &= 0.20 \\ 0.35\pi_1 &= 0.20 \\ \pi_1 &= 0.57 \end{aligned}$$

Other examples of Markov processes include the promotion of managers to various positions within an organization, the migration of people into and out of various regions of the country, and the progression of students through the years of college, including eventually dropping out or graduating.

and

$$\pi_2 = 1 - 0.57 = 0.43$$

We see that the proposed promotional strategy will increase Ashley's market share from $\pi_2 = 0.33$ to $\pi_2 = 0.43$. Suppose that the total market consists of 6000 customers per week. The new promotional strategy will increase the number of customers doing their weekly shopping at Ashley's from 2000 to 2580. If the average weekly profit per customer is \$10, the proposed promotional strategy can be expected to increase Ashley's profits by \$5800 per week. If the cost of the promotional campaign is less than \$5800 per week, Ashley's should consider implementing the strategy.

TABLE 17.4 REVISED TRANSITION PROBABILITIES FOR MURPHY'S AND ASHLEY'S GROCERY STORES

Current Weekly Shopping Period	Next Weekly Shopping Period: Murphy's Foodliner	Ashley's Supermarket
Murphy's Foodliner	0.85	0.15
Ashley's Supermarket	0.20	0.80

[2] Even though equations (17.2) and (17.3) provide two equations and two unknowns, we must include equation (17.4) when solving for π_1 and π_2 to ensure that the sum of steady-state probabilities will equal 1.

This example demonstrates how a Markov analysis of a firm's market share can be useful in decision making. Suppose that instead of trying to attract customers from Murphy's Foodliner, Ashley's directed a promotional effort at increasing the loyalty of its own customers. In this case, p_{22} would increase and p_{21} would decrease. Once we knew the amount of the change, we could calculate new steady-state probabilities and compute the impact on profits.

NOTES AND COMMENTS

1. The Markov processes presented in this section have what is called the memoryless property: the current state of the system together with the transition probabilities contain all the information necessary to predict the future behavior of the system. The prior states of the system do not have to be considered.
2. Analysis of a Markov process model is not intended to optimize any particular aspect of a system. Rather, the analysis predicts or describes the future and steady-state behavior of the system. For instance, in the grocery store example, the analysis of the steady-state behavior provided a forecast or prediction of the market shares for the two competitors. In other applications, quantitative analysts have extended the study of Markov processes to what are called *Markov decision processes*. In these models, decisions can be made at each period that affect the transition probabilities and hence influence the future behavior of the system. Markov decision processes have been used in analyzing machine breakdown and maintenance operations, determining drug treatment strategies for patients in hospitals, developing inspection strategies, evaluating managerial strategies in a baseball game, and analyzing financial investments.

17.2 Accounts Receivable Analysis

An accounting application in which Markov processes have produced useful results involves the estimation of the allowance for doubtful accounts receivable. This allowance is an estimate of the amount of accounts receivable that will ultimately prove to be uncollectible (i.e., bad debts).

Let us consider the accounts receivable situation for Heidman's Department Store. Heidman's uses two aging categories for its accounts receivable: (1) accounts that are classified as 0–30 days old, and (2) accounts that are classified as 31–90 days old. If any portion of an account balance exceeds 90 days, that portion is written off as a bad debt. Heidman's follows the procedure of aging the total balance in any customer's account according to the oldest unpaid bill. For example, suppose that one customer's account balance on September 30 is as follows:

Date of Purchase		Amount Charged
August 15		$25
September 18		10
September 28		50
	Total	$85

An aging of the total balance of accounts receivable performed on September 30 would assign the total balance of $85 to the 31–90-day category because the oldest unpaid bill of

August 15 is 46 days old. Let us assume that one week later, October 7, the customer pays the August 15 bill of $25. The remaining total balance of $60 would now be placed in the 0–30-day category since the oldest unpaid amount, corresponding to the September 18 purchase, is less than 31 days old. This method of aging accounts receivable is called the *total balance method* because the total account balance is placed in the age category corresponding to the oldest unpaid amount.

Note that under the total balance method of aging accounts receivable, dollars appearing in a 31–90-day category at one point in time may appear in a 0–30-day category at a later point in time. In the preceding example, this movement between categories was true for $60 of September billings, which shifted from a 31–90-day to a 0–30-day category after the August bill had been paid.

Let us assume that on December 31 Heidman's shows a total of $3000 in its accounts receivable and that the firm's management would like an estimate of how much of the $3000 will eventually be collected and how much will eventually result in bad debts. The estimated amount of bad debts will appear as an allowance for doubtful accounts in the year-end financial statements.

Let us see how we can view the accounts receivable operation as a Markov process. First, concentrate on what happens to *one* dollar currently in accounts receivable. As the firm continues to operate into the future, we can consider each week as a trial of a Markov process with a dollar existing in one of the following states of the system:

State 1. Paid category
State 2. Bad debt category
State 3. 0–30-day category
State 4. 31–90-day category

Thus, we can track the week-by-week status of one dollar by using a Markov analysis to identify the state of the system at a particular week or period.

Using a Markov process model with the preceding states, we define the transition probabilities as follows:

p_{ij} = probability of a dollar in state i in one week moving to state j in the next week

Based on historical transitions of accounts receivable dollars, the following matrix of transition probabilities, P, has been developed for Heidman's Department Store:

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.4 & 0.0 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 & 0.1 \end{bmatrix}$$

Note that the probability of a dollar in the 0–30-day category (state 3) moving to the paid category (state 1) in the next period is 0.4. Also, this dollar has a 0.3 probability it will remain in the 0–30-day category (state 3) one week later, and a 0.3 probability that it will be in the 31–90-day category (state 4) one week later. Note also that a dollar in a 0–30-day account cannot make the transition to a bad debt (state 2) in one week.

When absorbing states are present, each row of the transition matrix corresponding to an absorbing state will have a single 1 and all other probabilities will be 0.

An important property of the Markov process model for Heidman's accounts receivable situation is the presence of *absorbing states*. For example, once a dollar makes a transition to state 1, the paid state, the probability of making a transition to any other state is zero. Similarly, once a dollar is in state 2, the bad debt state, the probability of a transition to any other state is zero. Thus, once a dollar reaches state 1 or state 2, the system will remain in this state forever. We can conclude that all accounts receivable dollars will eventually be absorbed into either the paid or the bad debt state, and hence the name **absorbing state**.

Fundamental Matrix and Associated Calculations

Whenever a Markov process has absorbing states, the probability of remaining forever in a nonabsorbing state is zero, so we are solely interested in knowing the probability that a unit will eventually end up in each of the absorbing states. Due to the presence of absorbing states, steady-state probabilities independent of the initial state vector do not exist. For example, in the Heidman's Department Store problem, a dollar currently in state 1 (paid) will never leave state 1, and a dollar currently in state 2 (bad debt) will never leave state 2. However, we may want to know the probability that a dollar currently in the 0–30-day age category will end up paid (absorbing state 1) as well as the probability that a dollar in this age category will end up a bad debt (absorbing state 2). We also want to know these absorbing-state probabilities for a dollar currently in the 31–90-day age category.

The algebraic computation of the absorbing-state probabilities requires the determination and use of what is called a **fundamental matrix**. The mathematical logic underlying the fundamental matrix is beyond the scope of this text. However, as we show, the fundamental matrix is derived from the matrix of transition probabilities and is relatively easy to compute for Markov processes with a small number of states. In the following example, we show the computation of the fundamental matrix and the determination of the absorbing-state probabilities for Heidman's Department Store.

Absorbing-state probabilities can also be numerically computed by repeatedly multiplying the vector of state probabilities and the matrix of transition probabilities similar to the calculation in Section 17.1.

We begin the computations by partitioning the matrix of transition probabilities into the following four parts:

$$P = \left[\begin{array}{cc|cc} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ \hline 0.4 & 0.0 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 & 0.1 \end{array}\right] = \left[\begin{array}{cc|cc} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ \hline \multicolumn{2}{c|}{R} & \multicolumn{2}{c}{Q} \\ & & & \end{array}\right]$$

where

$$R = \begin{bmatrix} 0.4 & 0.0 \\ 0.4 & 0.2 \end{bmatrix} \qquad Q = \begin{bmatrix} 0.3 & 0.3 \\ 0.3 & 0.1 \end{bmatrix}$$

A matrix N, called a *fundamental matrix*, can be calculated using the following formula:

$$N = (I - Q)^{-1} \tag{17.5}$$

where I is an identity matrix with 1s on the main diagonal and 0s elsewhere. The superscript -1 is used to indicate the inverse of the matrix $(I - Q)$. In Appendix 17.1 we present formulas for finding the inverse of a matrix with two rows and two columns. In Appendix 17.2 we show how Excel's MINVERSE function can be used to compute an inverse.

Before proceeding, we note that to use equation (17.5), the identity matrix I must be chosen such that it has the *same size or dimensionality* as the matrix Q. In our example problem, Q has two rows and two columns, so we must choose

$$I = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

Let us now continue with the example problem by computing the fundamental matrix:

$$I - Q = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.3 \\ 0.3 & 0.1 \end{bmatrix}$$
$$= \begin{bmatrix} 0.7 & -0.3 \\ -0.3 & 0.9 \end{bmatrix}$$

and (see Appendix 17.1)

$$N = (I - Q)^{-1} = \begin{bmatrix} 1.67 & 0.56 \\ 0.56 & 1.30 \end{bmatrix}$$

If we multiply the fundamental matrix N times the R portion of the P matrix, we obtain the probabilities that accounts receivable dollars initially in states 3 or 4 will eventually reach each of the absorbing states. The multiplication of N times R for the Heidman's Department Store problem provides the following results (again, see Appendix 17.1 for the steps of this matrix multiplication):

$$NR = \begin{bmatrix} 1.67 & 0.56 \\ 0.56 & 1.30 \end{bmatrix} \begin{bmatrix} 0.4 & 0.0 \\ 0.4 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.89 & 0.11 \\ 0.74 & 0.26 \end{bmatrix}$$

The first row of the product NR is the probability that a dollar in the 0–30-day age category will end up in each absorbing state. Thus, we see a 0.89 probability that a dollar in the 0–30-day category will eventually be paid and a 0.11 probability that it will become a bad debt. Similarly, the second row shows the probabilities associated with a dollar in the 31–90-day category; that is, a dollar in the 31–90-day category has a 0.74 probability of eventually being paid and a 0.26 probability of proving to be uncollectible. Using this information, we can predict the amount of money that will be paid and the amount that will be lost as bad debts.

Establishing the Allowance for Doubtful Accounts

Let B represent a two-element vector that contains the current accounts receivable balances in the 0–30-day and the 31–90-day categories; that is,

$$B = [b_1 \quad b_2]$$

Total dollars in the 0–30-day category (b_1)

Total dollars in the 31–90-day category (b_2)

Suppose that the December 31 balance of accounts receivable for Heidman's shows \$1000 in the 0–30-day category (state 3) and \$2000 in the 31–90-day category (state 4).

$$B = [1000 \quad 2000]$$

We can multiply B times NR to determine how much of the \$3000 will be collected and how much will be lost. For example,

$$\begin{aligned} BNR &= [1000 \quad 2000] \begin{bmatrix} 0.89 & 0.11 \\ 0.74 & 0.26 \end{bmatrix} \\ &= [2370 \quad 630] \end{aligned}$$

Thus, we see that \$2370 of the accounts receivable balances will be collected and \$630 will be written off as a bad debt expense. Based on this analysis, the accounting department would set up an allowance for doubtful accounts of \$630.

The matrix multiplication of BNR is simply a convenient way of computing the eventual collections and bad debts of the accounts receivable. Recall that the NR matrix showed a 0.89 probability of collecting dollars in the 0–30-day category and a 0.74 probability of collecting dollars in the 31–90-day category. Thus, as was shown by the BNR calculation, we expect to collect a total of $(1000)0.89 + (2000)0.74 = 890 + 1480 = \$2370$.

Suppose that on the basis of the previous analysis Heidman's would like to investigate the possibility of reducing the amount of bad debts. Recall that the analysis indicated that 11% of the amount in the 0–30-day age category and 26% of the amount in the 31–90-day age category will prove to be uncollectible. Let us assume that Heidman's is considering instituting a new credit policy involving a discount for prompt payment.

Management believes that the policy under consideration will increase the probability of a transition from the 0–30-day age category to the paid category and decrease the probability of a transition from the 0–30-day to the 31–90-day age category. Let us assume that a careful study of the effects of this new policy leads management to conclude that the following transition matrix would be applicable:

$$P = \left[\begin{array}{cc|cc} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ \hline 0.6 & 0.0 & 0.3 & 0.1 \\ 0.4 & 0.2 & 0.3 & 0.1 \end{array}\right]$$

We see that the probability of a dollar in the 0–30-day age category making a transition to the paid category in the next period has increased to 0.6, and that the probability of a dollar in the 0–30-day age category making a transition to the 31–90-day category has decreased to 0.1. To determine the effect of these changes on bad debt expense, we must calculate N, NR, and BNR. We begin by using equation (17.5) to calculate the fundamental matrix N:

$$N = (I - Q)^{-1} = \left\{ \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.1 \\ 0.3 & 0.1 \end{bmatrix} \right\}^{-1}$$

$$= \begin{bmatrix} 0.7 & -0.1 \\ -0.3 & 0.9 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1.5 & 0.17 \\ 0.5 & 1.17 \end{bmatrix}$$

By multiplying N times R, we obtain the new probabilities that the dollars in each age category will end up in the two absorbing states:

$$NR = \begin{bmatrix} 1.5 & 0.17 \\ 0.5 & 1.17 \end{bmatrix} \begin{bmatrix} 0.6 & 0.0 \\ 0.4 & 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.97 & 0.03 \\ 0.77 & 0.23 \end{bmatrix}$$

We see that with the new credit policy we would expect only 3% of the funds in the 0–30-day age category and 23% of the funds in the 31–90-day age category to prove to be uncollectible. If, as before, we assume a current balance of \$1000 in the 0–30-day age category and \$2000 in the 31–90-day age category, we can calculate the total amount of accounts receivable that will end up in the two absorbing states by multiplying B times NR. We obtain

Problem 13, which provides a variation of Heidman's Department Store problem, will give you practice in analyzing Markov processes with absorbing states.

$$BNR = [1000 \quad 2000] \begin{bmatrix} 0.97 & 0.03 \\ 0.77 & 0.23 \end{bmatrix}$$

$$= [2510 \quad 490]$$

Thus, the new credit policy shows a bad debt expense of \$490. Under the previous credit policy, we found the bad debt expense to be \$630. Thus, a savings of \$630 – \$490 = \$140 could be expected as a result of the new credit policy. Given the total accounts receivable balance of \$3000, this savings represents a $\$140/\$3000 = 4.7\%$ reduction in bad debt expense.

After considering the costs involved, management can evaluate the economics of adopting the new credit policy. If the cost, including discounts, is less than 4.7% of the accounts receivable balance, we would expect the new policy to lead to increased profits for Heidman's Department Store.

Summary

In this chapter we presented Markov process models as well as examples of their application. We saw that a Markov analysis could provide helpful decision-making information about a situation that involves a sequence of repeated trials with a finite number of possible states on each trial. A primary objective is obtaining information about the probability of each state after a large number of transitions or time periods.

A market share application showed the computational procedure for determining the steady-state probabilities that could be interpreted as market shares for two competing supermarkets. In an accounts receivable application, we introduced the notion of absorbing states; for the two absorbing states, referred to as the paid and bad debt categories, we showed how to determine the percentage of an accounts receivable balance that would be absorbed in each of these states.

Markov process models have also been used to analyze strategies in sporting events. The Q.M. in Action, Markov Processes and Canadian Curling, describes the advantage gained in the sport of curling from winning the opening coin toss.

Q.M. *in* ACTION

*MARKOV PROCESSES AND CANADIAN CURLING**

Curling is a sport played on a strip of ice 14 feet wide and 146 feet long—about half the length of a football field. At the end of each strip is a "house" composed of four concentric circles etched in the ice, much like the target in a dartboard. The object is to slide a curling stone—called a rock—down the strip of ice and have it finish as close to the center of the house (the bulls-eye) as possible. A game consists of 10 ends. In an end, each team slides eight rocks down the strip and then the score is tallied. The team with the rock closest to the center of the house wins one or more points. A point is scored for every rock inside the closest rock for the other team. No rocks in the house means no score for the end.

The team that goes last has an advantage. For instance, that team has the opportunity to execute a "take out" by knocking the other team's rock(s) out of the house with its last shot. The team that goes last in an end is said to have the hammer. At the beginning of the game a coin toss determines which team starts with the hammer. As the game progresses, the hammer switches sides after any end in which the team with the hammer scores. If no score is made in an end, the hammer does not switch sides.

A Markov model was developed to determine the expected value of winning the coin toss to start the game with the hammer. Data were obtained for 8421 games played in the Canadian Men's Curling Championship over the 13 years from 1985 to 1997. The transition probabilities were based on the probability distributions for points scored in each of the 10 ends. An interesting finding was that the transition probabilities for the first end and the last end (and any extra ends) differed from those for the middle ends (ends 2 through 9).

Results of the Markov analysis showed that the expected score differential in favor of the team winning the opening coin toss was 1.115 when using three separate sets of transition probabilities. When one set of aggregate transition probabilities was used for all ends, the expected score differential in favor of the team winning the opening toss was 1.006. These results clearly indicate a significant advantage in winning the opening toss.

*Based on Kent J. Kostuk and Keith A. Willoughby, "OR/MS 'Rocks' the 'House,'" *OR/MS Today* (December 1999): 36–39.

Glossary

Markov chain with stationary transition probabilities A Markov process where the transition probabilities remain constant over time and the probability of being in a particular state at any one time period depends only on the state in the immediately preceding time period.
Trials of the process The events that trigger transitions of the system from one state to another. In many applications, successive time periods represent the trials of the process.
State of the system The condition of the system at any particular trial or time period.
Transition probability Given that the system is in state i during one period, the transition probability p_{ij} is the probability that the system will be in state j during the next period.
State probability The probability that the system will be in any particular state. (That is, $\pi_i(n)$ is the probability of the system being in state i in period n.)
Steady-state probability The probability that the system will be in any particular state after a large number of transitions. Once steady state has been reached, the state probabilities do not change from period to period.
Absorbing state A state is said to be absorbing if the probability of making a transition out of that state is zero. Thus, once the system has made a transition into an absorbing state, it will remain there.
Fundamental matrix A matrix necessary for the computation of probabilities associated with absorbing states of a Markov process.

Problems

1. In the market share analysis of Section 17.1, suppose that we are considering the Markov process associated with the shopping trips of one customer, but we do not know where the customer shopped during the last week. Thus, we might assume a 0.5 probability that the customer shopped at Murphy's and a 0.5 probability that the customer shopped at Ashley's at period 0; that is, $\pi_1(0) = 0.5$ and $\pi_2(0) = 0.5$. Given these initial state probabilities, develop a table similar to Table 17.2, showing the probability of each state in future periods. What do you observe about the long-run probabilities of each state?
2. Management of the New Fangled Softdrink Company believes that the probability of a customer purchasing Red Pop or the company's major competition, Super Cola, is based on the customer's most recent purchase. Suppose that the following transition probabilities are appropriate:

	To	
From	**Red Pop**	**Super Cola**
Red Pop	0.9	0.1
Super Cola	0.1	0.9

   a. Show the two-period tree diagram for a customer who last purchased Red Pop. What is the probability that this customer purchases Red Pop on the second purchase?
   b. What is the long-run market share for each of these two products?
   c. A Red Pop advertising campaign is being planned to increase the probability of attracting Super Cola customers. Management believes that the new campaign will increase to 0.15 the probability of a customer switching from Super Cola to Red Pop. What is the projected effect of the advertising campaign on the market shares?

SELF test

3. The email server at Rockbottom University has been experiencing downtime. Let us assume that the trials of an associated Markov process are defined as one-hour periods and that the

probability of the system being in a running state or a down state is based on the state of the system in the previous period. Historical data show the following transition probabilities:

	To	
From	**Running**	**Down**
Running	0.90	0.10
Down	0.30	0.70

a. If the system is initially running, what is the probability of the system being down in the next hour of operation?
b. What are the steady-state probabilities of the system being in the running state and in the down state?

4. One cause of the downtime in Problem 3 was traced to a specific piece of computer hardware. Management believes that switching to a different hardware component will result in the following transition probabilities:

	To	
From	**Running**	**Down**
Running	0.95	0.05
Down	0.60	0.40

a. What are the steady-state probabilities of the system being in the running and down states?
b. If the cost of the system being down for any period is estimated to be $500 (including lost productivity and maintenance costs), what is the breakeven cost for the new hardware component on a time-period basis?

5. A major traffic problem in the Greater Cincinnati area involves traffic attempting to cross the Ohio River from Cincinnati to Kentucky using Interstate 75. Let us assume that the probability of no traffic delay in one period, given no traffic delay in the preceding period, is 0.85 and that the probability of finding a traffic delay in one period, given a delay in the preceding period, is 0.75. Traffic is classified as having either a delay or a no-delay state, and the period considered is 30 minutes.
a. Assume that you are a motorist entering the traffic system and receive a radio report of a traffic delay. What is the probability that for the next 60 minutes (two time periods) the system will be in the delay state? Note that this result is the probability of being in the delay state for two consecutive periods.
b. What is the probability that in the long run the traffic will not be in the delay state?
c. An important assumption of the Markov process models presented in this chapter has been the constant or stationary transition probabilities as the system operates in the future. Do you believe this assumption should be questioned for this traffic problem? Explain.

6. Rock-Paper-Scissors is a simple game in which two players compete by simultaneously choosing rock, paper, or scissors. According to the classic rules of the game, paper beats rock, scissors beats paper, and rock beats scissors. After watching one particular player compete in this game for many rounds, you have observed that she chooses rock, paper, or scissors based on her previous choice according to the following transition probabilities:

		Next Choice	
Previous Choice	**Rock**	**Paper**	**Scissors**
Rock	0.27	0.42	0.31
Paper	0.36	0.15	0.49
Scissors	0.18	0.55	0.27

Assume you are competing against this player and that you must decide on all future choices now (before seeing any additional opponent choices).

a. Given the opposing player last chose Rock, what is your best choice in the next round as her opponent?
b. Given the opposing player last chose Rock, show the two-period tree diagram.
c. Calculate $\Pi(2)$ and use this to find the probability that the opposing player will choose Paper in the second round.

7. Data collected from selected major metropolitan areas in the eastern United States show that 2% of individuals living within the city limits move to the suburbs during a one-year period while 1% of individuals living in the suburbs move to the city during a one-year period. Answer the following questions, assuming that this process is modeled by a Markov process with two states: city and suburbs:
   a. Prepare the matrix of transition probabilities.
   b. Compute the steady-state probabilities.
   c. In a particular metropolitan area, 40% of the population lives in the city, and 60% of the population lives in the suburbs. What population changes do your steady-state probabilities project for this metropolitan area?

SELF test

8. Assume that a third grocery store, Quick Stop Groceries, enters the market share and customer loyalty situation described in Section 17.1. Quick Stop Groceries is smaller than either Murphy's Foodliner or Ashley's Supermarket. However, Quick Stop's convenience with faster service and gasoline for automobiles can be expected to attract some customers who currently make weekly shopping visits to either Murphy's or Ashley's. Assume that the transition probabilities are as follows:

	To		
From	**Murphy's**	**Ashley's**	**Quick Stop**
Murphy's Foodliner	0.85	0.10	0.05
Ashley's Supermarket	0.20	0.75	0.05
Quick Stop Groceries	0.15	0.10	0.75

   a. Compute the steady-state probabilities for this three-state Markov process.
   b. What market share will Quick Stop obtain?
   c. With 1000 customers, the original two-state Markov process in Section 17.1 projected 667 weekly customer trips to Murphy's Foodliner and 333 weekly customer trips to Ashley's Supermarket. What impact will Quick Stop have on the customer visits at Murphy's and Ashley's? Explain.

9. The purchase patterns for two brands of toothpaste can be expressed as a Markov process with the following transition probabilities:

	To	
From	**Special B**	**MDA**
Special B	0.90	0.10
MDA	0.05	0.95

   a. Which brand appears to have the most loyal customers? Explain.
   b. What are the projected market shares for the two brands?

10. Suppose that in Problem 9 a new toothpaste brand enters the market such that the following transition probabilities exist:

	To		
From	**Special B**	**MDA**	**T-White**
Special B	0.80	0.10	0.10
MDA	0.05	0.75	0.20
T-White	0.40	0.30	0.30

What are the new long-run market shares? Which brand will suffer most from the introduction of the new brand of toothpaste?

11. In American football, touchdowns are worth 6 points. After scoring a touchdown, the scoring team may subsequently attempt to score one or two additional points. Going for one point is virtually an assured success, while going for two points is successful only with probability p. Consider the following game situation. The Temple Wildcats are losing by 14 points to the Killeen Tigers near the end of regulation time. The only way for Temple to win (or tie) this game is to score two touchdowns while not allowing Killeen to score again. The Temple coach must decide whether to attempt a 1-point or 2-point conversion after each touchdown. If the score is tied at the end of regulation time, the game goes into overtime where the first team to score wins. The Temple coach believes that there is a 50% chance that Temple will win if the game goes into overtime. The probability of successfully converting a 1-point conversion is 1.0. The probability of successfully converting a 2-point conversion is p.
    a. Assume Temple will score two touchdowns and Killeen will not score. Define the state of the system as the number of points by which Temple is currently trailing. Create a tree diagram for the situation in which Temple's coach attempts a 2-point conversion after the first touchdown. If the 2-point conversion is successful, Temple will go for 1 point after the second touchdown to win the game. If the 2-point conversion is unsuccessful, Temple will go for 2 points after the second touchdown in an attempt to tie the game and go to overtime.
    b. Create the transition probability matrix for this decision problem.
    c. If Temple's coach goes for a 1-point conversion after each touchdown, the game is assured of going to overtime and Temple will win with probability 0.5. For what values of p is the strategy defined in part a superior to going for 1 point after each touchdown?

12. Given the following transition matrix with states 1 and 2 as absorbing states, what is the probability that units in states 3 and 4 end up in each of the absorbing states?

$$P = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.2 & 0.1 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.1 & 0.5 \end{bmatrix}$$

SELF test

13. In the Heidman's Department Store problem of Section 17.2, suppose that the following transition matrix is appropriate:

$$P = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.25 & 0.25 \\ 0.5 & 0.2 & 0.05 & 0.25 \end{bmatrix}$$

If Heidman's has \$4000 in the 0–30-day category and \$5000 in the 31–90-day category, what is your estimate of the amount of bad debts the company will experience?

14. The KLM Christmas Tree Farm owns a plot of land with 5000 evergreen trees. Each year KLM allows retailers of Christmas trees to select and cut trees for sale to individual customers. KLM protects small trees (usually less than 4 feet tall) so that they will be available for sale in future years. Currently, 1500 trees are classified as protected trees, while the remaining 3500 are available for cutting. However, even though a tree is available for cutting in a given year, it may not be selected for cutting until future years. Most trees not cut in a given year live until the next year, but some diseased trees are lost every year.

In viewing the KLM Christmas tree operation as a Markov process with yearly periods, we define the following four states:

State 1. Cut and sold

State 2. Lost to disease

State 3. Too small for cutting

State 4. Available for cutting but not cut and sold

The following transition matrix is appropriate:

$$P = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.1 & 0.2 & 0.5 & 0.2 \\ 0.4 & 0.1 & 0.0 & 0.5 \end{bmatrix}$$

How many of the farm's 5000 trees will be sold eventually, and how many will be lost?

15. A large corporation collected data on the reasons both middle managers and senior managers leave the company. Some managers eventually retire, but others leave the company prior to retirement for personal reasons, including more attractive positions with other firms. Assume that the following matrix of one-year transition probabilities applies with the four states of the Markov process being retirement, leaves prior to retirement for personal reasons, stays as a middle manager, and stays as a senior manager.

	Retirement	**Leaves—Personal**	**Middle Manager**	**Senior Manager**
Retirement	1.00	0.00	0.00	0.00
Leaves—Personal	0.00	1.00	0.00	0.00
Middle Manager	0.03	0.07	0.80	0.10
Senior Manager	0.08	0.01	0.03	0.88

a. What states are considered absorbing states? Why?
b. Interpret the transition probabilities for the middle managers.
c. Interpret the transition probabilities for the senior managers.
d. What percentage of the current middle managers will eventually retire from the company? What percentage will leave the company for personal reasons?
e. The company currently has 920 managers: 640 middle managers and 280 senior managers. How many of these managers will eventually retire from the company? How many will leave the company for personal reasons?

16. Players in a particular sports league are classified based on the amount of time they play as backups or starters. Players can also become seriously injured or they can retire from competition. Consider the following transition probability matrix.

	Backup	Starter	Injured	Retired
Backup	0.4	0.4	0.1	0.1
Starter	0.1	0.5	0.15	0.25
Injured	0	0	1	0
Retired	0	0	0	1

a. What state(s) are absorbing states?
b. What percentage of current starters will eventually be injured?
c. Currently a team called The Sharks has eight backups and five starters. How many of the players do you expect to end up injured? Retired?

Case Problem Dealer's Absorbing State Probabilities in Blackjack

The game of blackjack (sometimes called "21") is a popular casino game. The goal is to have a hand with a value of 21 or as close to 21 as possible without exceeding 21. The player and the dealer are each dealt two cards initially. Both the player and dealer may draw additional cards (called "taking a hit") in order to improve their hand. If either the player or dealer takes a hit and the value of the hand exceeds 21, the player or dealer is said to have gone broke and loses. Face cards and tens count 10 points, aces can be counted as 1 or 11, and all other cards count at their face value. The dealer's advantage is that the player must decide on whether to take a hit first. The player who takes a hit and goes over 21 goes broke and loses, even if the dealer later goes broke. For instance, if the player has 16 and draws any card with a value higher than a 5, the player goes broke and loses. For this reason, players will often decide not to take a hit when the value of their hand is 12 or greater.

The dealer's hand is dealt with one card up and one card down. So, the player's decision of whether to take a hit is based on knowledge of the dealer's up card. A gambling professional asks you to help determine the probability of the ending value of the dealer's hand given different up cards. House rules at casinos require that the dealer continue to take a hit until the dealer's hand reaches a value of 17 or higher. Having just studied Markov processes, you suggest that the dealer's process of taking hits can be modeled as a Markov process with absorbing states.

Managerial Report

Prepare a report for the professional gambler that summarizes your findings. Include the following:

1. At some casinos, the dealer is required to stay (stop taking hits) when the dealer hand reaches soft or hard 17. A hand of soft 17 is one including an ace that may be counted as 1 or 11. In all casinos, the dealer is required to stay with soft 18, 19, 20, or 21. For each possible up card, determine the probability that the ending value of the dealer's hand is 17, 18, 19, 20, 21, or broke.
2. At other casinos, the dealer is required to take a hit on soft 17, but must stay on all other hands with a value of 17, 18, 19, 20, or 21. For this situation, determine the probability of the ending value of the dealer's hand.
3. Comment on whether the house rule of staying on soft 17 or hitting on soft 17 appears better for the player.

Appendix 17.1 Matrix Notation and Operations

Matrix Notation

A *matrix* is a rectangular arrangement of numbers. For example, consider the following matrix that we have named D:

$$D = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 5 \end{bmatrix}$$

The matrix D is said to consist of six elements, where each element of D is a number. To identify a particular element of a matrix, we have to specify its location. Therefore, we introduce the concepts of rows and columns.

All elements across some horizontal line in a matrix are said to be in a row of the matrix. For example, elements 1, 3, and 2 in D are in the first row, and elements 0, 4, and 5 are in the second row. By convention, we refer to the top row as row 1, the second row from the top as row 2, and so on.

All elements along some vertical line are said to be in a column of the matrix. Elements 1 and 0 in D are elements in the first column, elements 3 and 4 are elements of the second column, and elements 2 and 5 are elements of the third column. By convention, we refer to the leftmost column as column 1, the next column to the right as column 2, and so on.

We can identify a particular element in a matrix by specifying its row and column position. For example, the element in row 1 and column 2 of D is the number 3. This position is written as

$$d_{12} = 3$$

In general, we use the following notation to refer to the specific elements of D:

$$d_{ij} = \text{element located in the } i\text{th row and } j\text{th column of } D$$

We always use capital letters for the names of matrixes and the corresponding lowercase letters with two subscripts to denote the elements.

The *size* of a matrix is the number of rows and columns in the matrix and is written as the number of rows $\times$ the number of columns. Thus, the size of D is 2×3.

Frequently we will encounter matrixes that have only one row or one column. For example,

$$G = \begin{bmatrix} 6 \\ 4 \\ 2 \\ 3 \end{bmatrix}$$

is a matrix that has only one column. Whenever a matrix has only one column, we call the matrix a *column vector*. In a similar manner, any matrix that has only one row is called a *row vector*. Using our previous notation for the elements of a matrix, we could refer to specific elements in G by writing g_{ij}. However, since G has only one column, the column position is unimportant, and we need only specify the row the element of interest is in. That is, instead of referring to elements in a vector using g_{ij}, we specify only one subscript, which denotes the position of the element in the vector. For example,

$$g_1 = 6 \qquad g_2 = 4 \qquad g_3 = 2 \qquad g_4 = 3$$

Matrix Operations

Matrix Transpose The transpose of a matrix is formed by making the rows in the original matrix the columns in the transpose matrix, and by making the columns in the original matrix the rows in the transpose matrix. For example, the transpose of the matrix

$$D = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 5 \end{bmatrix}$$

is

$$D^t = \begin{bmatrix} 1 & 0 \\ 3 & 4 \\ 2 & 5 \end{bmatrix}$$

Note that we use the superscript t to denote the transpose of a matrix.

Matrix Multiplication We demonstrate how to perform two types of matrix multiplication: (1) multiplying two vectors, and (2) multiplying a matrix times a matrix.

The product of a row vector of size $1 \times n$ times a column vector of size $n \times 1$ is the number obtained by multiplying the first element in the row vector times the first element in the column vector, the second element in the row vector times the second element in the column vector, continuing on through the last element in the row vector times the last element in the column vector, and then summing the products. Suppose, for example, that we wanted to multiply the row vector H times the column vector G, where

$$H = [2 \quad 1 \quad 5 \quad 0] \text{ and } G = \begin{bmatrix} 6 \\ 4 \\ 2 \\ 3 \end{bmatrix}$$

The product HG, referred to as a vector product, is given by

$$HG = 2(6) + 1(4) + 5(2) + 0(3) = 26$$

The product of a matrix of size $p \times n$ and a matrix of size $n \times m$ is a new matrix of size $p \times m$. The element in the ith row and jth column of the new matrix is given by the vector product of the ith row of the $p \times n$ matrix times the jth column of the $n \times m$ matrix. Suppose, for example, that we want to multiply D times A, where

$$D = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 5 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 0 & 4 \\ 1 & 5 & 2 \end{bmatrix}$$

Let $C = DA$ denote the product of D times A. The element in row 1 and column 1 of C is given by the vector product of the first row of D times the first column of A. Thus,

$$c_{11} = [1 \quad 3 \quad 2] \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 1(1) + 3(2) + 2(1) = 9$$

The element in row 2 and column 1 of C is given by the vector product of the second row of D times the first column of A. Thus,

$$c_{21} = [0 \quad 4 \quad 5] \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 0(1) + 4(2) + 5(1) = 13$$

Calculating the remaining elements of C in a similar fashion, we obtain

$$C = \begin{bmatrix} 9 & 13 & 21 \\ 13 & 25 & 26 \end{bmatrix}$$

Clearly, the product of a matrix and a vector is just a special case of multiplying a matrix times a matrix. For example, the product of a matrix of size $m \times n$ and a vector of size $n \times 1$ is a new vector of size $m \times 1$. The element in the ith position of the new vector is given by the vector product of the ith row of the $m \times n$ matrix times the $n \times 1$ column vector. Suppose, for example, that we want to multiply D times K, where

$$D = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 5 \end{bmatrix} \quad K = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

The first element of DK is given by the vector product of the first row of D times K. Thus,

$$[1 \quad 3 \quad 2]\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = 1(1) + 3(4) + 2(2) = 17$$

The second element of DK is given by the vector product of the second row of D and K. Thus,

$$[0 \quad 4 \quad 5]\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = 0(1) + 4(4) + 5(2) = 26$$

Hence, we see that the product of the matrix D times the vector K is given by

$$DK = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 17 \\ 26 \end{bmatrix}$$

Can any two matrixes be multiplied? The answer is no. To multiply two matrixes, the number of the columns in the first matrix must equal the number of rows in the second. If this property is satisfied, the matrixes are said to *conform for multiplication.* Thus, in our example, D and K could be multiplied because D had three columns and K had three rows.

Matrix Inverse The inverse of a matrix A is another matrix, denoted A^{-1}, such that $A^{-1}A = \mathrm{I}$ and $AA^{-1} = \mathrm{I}$. The inverse of any square matrix A consisting of two rows and two columns is computed as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} a_{22}/d & -a_{12}/d \\ -a_{21}/d & a_{11}/d \end{bmatrix}$$

where $d = a_{11}a_{22} - a_{21}a_{12}$ is the determinant of the 2×2 matrix A. For example, if

$$A = \begin{bmatrix} 0.7 & -0.3 \\ -0.3 & 0.9 \end{bmatrix}$$

then

$$d = (0.7)(0.9) - (-0.3)(-0.3) = 0.54$$

and

$$A^{-1} = \begin{bmatrix} 0.9/0.54 & 0.3/0.54 \\ 0.3/0.54 & 0.7/0.54 \end{bmatrix} = \begin{bmatrix} 1.67 & 0.56 \\ 0.56 & 1.30 \end{bmatrix}$$

Appendix 17.2 Matrix Inversion with Excel

Excel provides a function called MINVERSE that can be used to compute the inverse of a matrix. This function is extremely useful when the inverse of a matrix of size 3×3 or larger is desired. To see how it is used, suppose we want to invert the following 3×3 matrix:

$$\begin{bmatrix} 3 & 5 & 0 \\ 0 & 1 & 1 \\ 8 & 5 & 0 \end{bmatrix}$$

Enter the matrix into cells B3:D5 of an Excel worksheet. The following steps will compute the inverse and place it in cells B7:D9:

Step 1. Select cells **B7:D9**
Step 2. Type **=MINVERSE(B3:D5)**
Step 3. Press **Ctrl + Shift + Enter**

Step 3 may appear strange. Excel's MINVERSE function returns an array (matrix) and must be used in what Excel calls an array formula. In step 3, we must press the Ctrl and Shift keys while we press Enter. The inverse matrix will then appear in cells B7:D9. It is

$$\begin{bmatrix} -0.20 & 0 & 0.20 \\ 0.32 & 0 & -0.12 \\ -0.32 & 1 & 0.12 \end{bmatrix}$$