# CHAPTER 5

# Utility and Game Theory

**CONTENTS**

The decision analysis situations presented in Chapter 4 often expressed consequences or payoffs in terms of monetary values. With probability information available about the outcomes of the chance events, we defined the optimal decision alternative as the one that provided the best expected monetary value. However, in some situations the decision alternative with the best expected monetary value may not be the preferred alternative. A decision maker may also wish to consider intangible factors such as risk, image, or other nonmonetary criteria in order to evaluate the decision alternatives. When monetary value does not necessarily lead to the most preferred decision, expressing the value (or worth) of a consequence in terms of its utility will permit the use of expected utility to identify the most desirable decision alternative. The discussion of utility and its application in decision analysis is presented in the first part of this chapter.

In the last part of this chapter we introduce the topic of game theory. Game theory is the study of developing optimal strategies where two or more decision makers, usually called players, compete as adversaries. Game theory can be viewed as a relative of decision analysis. A key difference, however, is that each player selects a decision strategy not by considering the possible outcomes of a chance event, but by considering the possible strategies selected by one or more competing players.

We note here that utility and game theory are separable topics. Either or both may be studied, and it is not required that you cover one topic before the other. The QM in Action, Game Theory used in 700-MHZ Auction, describes how participants in an FCC auction used game theory to develop bidding strategies.

**Q.M. *in* ACTION**

*GAME THEORY USED IN 700-MHZ AUCTION**

On January 24, 2008 the Federal Communications Commission (FCC) auctioned the rights to operate the 700-MHz frequency band in the United States. This bandwidth became available due to the switch of over-the-air television broadcasts from analog to digital transmission. The 700-MHz frequency bandwidth is highly desirable to companies because the high frequency can penetrate walls and other obstacles. Companies including Google, AT&T, Verizon Wireless, and many others placed bids on the rights to operate in this frequency band.

Game theory was central to this auction, as it was used by the FCC to establish the overall rules and procedures for the auction. To promote competition, the FCC used a "blind auction" format in which each bid was anonymous. A blind auction assures that each bidder does not know which competitor(s) they are bidding against. Thus, large firms could not use their market dominance and deep pockets to intimidate smaller firms from placing additional bids. Further, bidding was allowed to continue until no new bids were received in order to prevent last-second winning bids (a practice known as auction sniping).

Most participants hired game theorists to devise bid strategies. Economists, mathematicians, engineers, and many others assisted companies in developing optimal bid plans. The auction lasted 261 rounds over 38 days and resulted in 101 winning bidders. The auction generated over $19 billion in revenue for the FCC.

*From E. Woyke, "Peeking into the Spectrum Auction," *Forbes*, 2007.

## 5.1 The Meaning of Utility

**Utility** is a measure of the total worth or relative desirability of a particular outcome; it reflects the decision maker's attitude toward a collection of factors such as profit, loss, and risk. Researchers have found that as long as the monetary value of payoffs stays within a range that the decision maker considers reasonable, selecting the decision alternative with the best expected monetary value usually leads to selection of the most preferred decision. However, when the payoffs are extreme, decision makers are often unsatisfied or uneasy with the decision that simply provides the best expected monetary value.

As an example of a situation in which utility can help in selecting the best decision alternative, let us consider the problem faced by Swofford, Inc., a relatively small real estate investment firm located in Atlanta, Georgia. Swofford currently has two investment opportunities that require approximately the same cash outlay. The cash requirements necessary prohibit Swofford from making more than one investment at this time. Consequently, three possible decision alternatives may be considered.

The three decision alternatives, denoted $d_1$, $d_2$, and $d_3$, are

$$d_1 = \text{make investment A}$$
$$d_2 = \text{make investment B}$$
$$d_3 = \text{do not invest}$$

The monetary payoffs associated with the investment opportunities depend on the investment decision and on the direction of the real estate market during the next six months (the chance event). Real estate prices will go up, remain stable, or go down. Thus the states of nature, denoted $s_1$, $s_2$, and $s_3$, are

$$s_1 = \text{real estate prices go up}$$
$$s_2 = \text{real estate prices remain stable}$$
$$s_3 = \text{real estate prices go down}$$

Using the best information available, Swofford has estimated the profits, or payoffs, associated with each decision alternative and state-of-nature combination. The resulting payoff table is shown in Table 5.1.

The best estimate of the probability that real estate prices will go up is 0.3; the best estimate of the probability that prices will remain stable is 0.5; and the best estimate of the probability that prices will go down is 0.2. Thus the expected values for the three decision alternatives are

$$\begin{aligned}
\text{EV}(d_1) &= 0.3(\$30{,}000) + 0.5(\$20{,}000) + 0.2(-\$50{,}000) = \$9000 \\
\text{EV}(d_2) &= 0.3(\$50{,}000) + 0.5(-\$20{,}000) + 0.2(-\$30{,}000) = -\$1000 \\
\text{EV}(d_3) &= 0.3(\$0) + 0.5(\$0) + 0.2(\$0) = \$0
\end{aligned}$$

Using the expected value approach, the optimal decision is to select investment A with an expected monetary value of \$9000. Is it really the best decision alternative? Let us consider

**TABLE 5.1** PAYOFF TABLE FOR SWOFFORD, INC.

| | State of Nature | | |
|---|---|---|---|
| **Decision Alternative** | **Prices Up $s_1$** | **Prices Stable $s_2$** | **Prices Down $s_3$** |
| **Investment A, $d_1$** | \$30,000 | \$20,000 | −\$50,000 |
| **Investment B, $d_2$** | \$50,000 | −\$20,000 | −\$30,000 |
| **Do not invest, $d_3$** | \$0 | \$0 | \$0 |

some other factors that relate to Swofford's capability for absorbing the loss of $50,000 if investment A is made and prices actually go down.

Actually, Swofford's current financial position is weak. This condition is partly reflected in Swofford's ability to make only one investment. More important, however, the firm's president believes that, if the next investment results in a substantial loss, Swofford's future will be in jeopardy. Although the expected value approach leads to a recommendation for $d_1$, do you think the firm's president would prefer this decision? We suspect that the president would select $d_2$ or $d_3$ to avoid the possibility of incurring a $50,000 loss. In fact, a reasonable conclusion is that, if a loss of even $30,000 could drive Swofford out of business, the president would select $d_3$, believing that both investments A and B are too risky for Swofford's current financial position.

The way we resolve Swofford's dilemma is first to determine Swofford's utility for the various monetary outcomes. Recall that the utility of any outcome is the total worth of that outcome, taking into account all risks and consequences involved. If the utilities for the various consequences are assessed correctly, the decision alternative with the highest expected utility is the most preferred, or best, alternative. In the next section we show how to determine the utility of the monetary outcomes so that the alternative with the highest expected utility can be identified.

## 5.2 Utility and Decision Making

*Utility values of 0 and 1 could have been selected here; we selected 0 and 10 in order to avoid any possible confusion between the utility value for a payoff and the probability p.*

The procedure we use to establish utility values for the payoffs in Swofford's situation requires that we first assign a utility value to the best and worst possible payoffs. Any values will work as long as the utility assigned to the best payoff is greater than the utility assigned to the worst payoff. In this case, $50,000 is the best payoff and −$50,000 is the worst. Suppose, then, that we arbitrarily make assignments to these two payoffs as follows:

$$\text{Utility of } -\$50{,}000 = U(-\$50{,}000) = 0$$
$$\text{Utility of } \$50{,}000 = U(\$50{,}000) = 10$$

Let us now determine the utility associated with every other payoff.

Consider the process of establishing the utility of a payoff of $30,000. First we ask Swofford's president to state a preference between a guaranteed $30,000 payoff and an opportunity to engage in the following **lottery**, or bet, for some probability of $p$ that we select:

Lottery: Swofford obtains a payoff of $50,000 with probability $p$
and a payoff of −$50,000 with probability $(1 - p)$

*p is often referred to as the indifference probability.*

If $p$ is very close to 1, Swofford's president would obviously prefer the lottery to the guaranteed payoff of $30,000 because the firm would virtually ensure itself a payoff of $50,000. If $p$ is very close to 0, Swofford's president would clearly prefer the guarantee of $30,000. In any event, as $p$ increases continuously from 0 to 1, the preference for the guaranteed payoff of $30,000 decreases and at some point is equal to the preference for the lottery. At this value of $p$, Swofford's president would have equal preference for the guaranteed payoff of $30,000 and the lottery; at greater values of $p$, Swofford's president would prefer the lottery to the guaranteed $30,000 payoff. For example, let us assume that when $p = 0.95$, Swofford's president is indifferent between the guaranteed payoff of \$30,000 and the lottery. For this value of $p$, we can compute the utility of a \$30,000 payoff as follows:

$$\begin{aligned} U(\$30{,}000) &= pU(\$50{,}000) + (1 - p)U(-\$50{,}000) \\ &= 0.95(10) + (0.05)(0) \\ &= 9.5 \end{aligned}$$

Obviously, if we had started with a different assignment of utilities for a payoff of \$50,000 and −\$50,000, the result would have been a different utility for \$30,000. For example, if we had started with an assignment of 100 for \$50,000 and 10 for −\$50,000, the utility of a \$30,000 payoff would be

$$\begin{aligned} U(\$30{,}000) &= 0.95(100) + 0.05(10) \\ &= 0.95 + 0.05 \\ &= 95.5 \end{aligned}$$

Hence, we must conclude that the utility assigned to each payoff is not unique but merely depends on the initial choice of utilities for the best and worst payoffs. We will discuss utility choice further at the end of the section. For now, however, we will continue to use a value of 10 for the utility of \$50,000 and a value of 0 for the utility of −\$50,000.

Before computing the utility for the other payoffs, let us consider the implication of Swofford's president assigning a utility of 9.5 to a payoff of \$30,000. Clearly, when $p = 0.95$, the expected value of the lottery is

$$\begin{aligned} \text{EV(lottery)} &= 0.95(\$50{,}000) + 0.05(-\$50{,}000) \\ &= \$47{,}500 - \$2{,}500 \\ &= \$45{,}000 \end{aligned}$$

Although the expected value of the lottery when $p = 0.95$ is \$45,000, Swofford's president is indifferent between the lottery (and its associated risk) and a guaranteed payoff of \$30,000. Thus, Swofford's president is taking a conservative, or risk-avoiding, viewpoint. A decision maker who would choose a guaranteed payoff over a lottery with a superior expected payoff is a risk avoider (or is said to be risk averse). The president would rather have \$30,000 for certain than risk anything greater than a 5% chance of incurring a loss of \$50,000. In other words, the difference between the EV of \$45,000 and the guaranteed payoff of \$30,000 is the risk premium that Swofford's president would be willing to pay to avoid the 5% chance of losing \$50,000.

*The difference between the expected value of the lottery and the guaranteed payoff can be viewed as the risk premium the decision maker is willing to pay.*

To compute the utility associated with a payoff of −\$20,000, we must ask Swofford's president to state a preference between a guaranteed −\$20,000 payoff and an opportunity to engage again in the following lottery:

Lottery: Swofford obtains a payoff of \$50,000 with probability $p$
and a payoff of −\$50,000 with probability $(1 - p)$

Note that this lottery is exactly the same as the one we used to establish the utility of a payoff of \$30,000 (in fact, we can use this lottery to establish the utility for any monetary value in the Swofford payoff table). We need to determine the value of $p$ that would make the president indifferent between a guaranteed payoff of −\$20,000 and the lottery. For example, we might begin by asking the president to choose between a certain loss of \$20,000 and the lottery with a payoff of \$50,000 with probability $p = 0.90$ and a payoff of −\$50,000 with probability $(1 - p) = 0.10$. What answer do you think we would get? Surely, with this high probability of obtaining a payoff of \$50,000, the president would elect the lottery. Next, we might ask whether $p = 0.85$ would result in indifference between the loss of \$20,000 for certain and the lottery. Again the president might prefer the lottery. Suppose that we continue until we get to $p = 0.55$, at which point the president is indifferent between the payoff of −\$20,000 and the lottery. That is, for any value of $p$ less than 0.55, the president would take a loss of \$20,000 for certain rather than risk the potential loss of \$50,000 with the lottery; and for any value of $p$ above 0.55, the president would choose the lottery. Thus, the utility assigned to a payoff of −\$20,000 is

$$\begin{aligned} U(-\$20{,}000) &= pU(\$50{,}000) + (1 - p)U(-\$50{,}000) \\ &= 0.55(10) + 0.45(0) \\ &= 5.5 \end{aligned}$$

Again let us assess the implication of this assignment by comparing it to the expected value approach. When $p = 0.55$, the expected value of the lottery is

$$\begin{aligned} \text{EV(lottery)} &= 0.55(\$50{,}000) + 0.45(-\$50{,}000) \\ &= \$27{,}500 - \$22{,}500 \\ &= \$5000 \end{aligned}$$

Thus, Swofford's president would just as soon absorb a certain loss of \$20,000 as take the lottery and its associated risk, even though the expected value of the lottery is \$5000. Once again this preference demonstrates the conservative, or risk-avoiding, point of view of Swofford's president.

In these two examples we computed the utility for the monetary payoffs of \$30,000 and −\$20,000. We can determine the utility for any monetary payoff $M$ in a similar fashion. First, we must find the probability $p$ for which the decision maker is indifferent between a guaranteed payoff of $M$ and a lottery with a payoff of \$50,000 with probability $p$ and −\$50,000 with probability $(1 - p)$. The utility of $M$ is then computed as follows:

$$\begin{aligned} U(M) &= pU(\$50{,}000) + (1 - p)U(-\$50{,}000) \\ &= p(10) + (1 - p)0 \\ &= 10p \end{aligned}$$

Using this procedure, we developed utility values for the rest of the payoffs in Swofford's problem. The results are presented in Table 5.2.

Now that we have determined the utility value of each of the possible monetary values, we can write the original payoff table in terms of utility values. Table 5.3 shows the utility for the various outcomes in the Swofford problem. The notation we use for the entries in the utility table is $U_{ij}$, which denotes the utility associated with decision alternative $d_i$ and state of nature $s_j$. Using this notation, we see that $U_{23} = 4.0$.

## The Expected Utility Approach

We can now apply the expected value computations introduced in Chapter 4 to the utilities in Table 5.3 in order to select an optimal decision alternative for Swofford, Inc. However, because utility values represent such a special case of expected value, we will refer to the

**TABLE 5.2** UTILITY OF MONETARY PAYOFFS FOR SWOFFORD, INC.

| Monetary Value | Indifference Value of $p$ | Utility Value |
|---|---|---|
| \$50,000 | Does not apply | 10.0 |
| \$30,000 | 0.95 | 9.5 |
| \$20,000 | 0.90 | 9.0 |
| \$ 0 | 0.75 | 7.5 |
| −\$20,000 | 0.55 | 5.5 |
| −\$30,000 | 0.40 | 4.0 |
| −\$50,000 | Does not apply | 0.0 |



**TABLE 5.3** UTILITY TABLE FOR SWOFFORD, INC.

| | State of Nature | | |
|---|---|---|---|
| **Decision Alternative** | **Prices Up $s_1$** | **Prices Stable $s_2$** | **Prices Down $s_3$** |
| **Investment A, $d_1$** | 9.5 | 9.0 | 0.0 |
| **Investment B, $d_2$** | 10.0 | 5.5 | 4.0 |
| **Do not invest, $d_3$** | 7.5 | 7.5 | 7.5 |



expected value when applied to utility values as the **expected utility (EU)**. Thus, the expected utility approach requires the analyst to compute the expected utility for each decision alternative and then select the alternative yielding the highest expected utility. With $N$ possible states of nature, the expected utility of a decision alternative $d_i$ is given by

$$\text{EU}(d_i) = \sum_{j=1}^{N} P(s_j)U_{ij} \tag{5.1}$$

The expected utility for each of the decision alternatives in the Swofford problem is

$$\text{EU}(d_1) = 0.3(9.5) + 0.5(9.0) + 0.2(0.0) = 7.35$$
$$\text{EU}(d_2) = 0.3(10.0) + 0.5(5.5) + 0.2(4.0) = 6.55$$
$$\text{EU}(d_3) = 0.3(7.5) + 0.5(7.5) + 0.2(7.5) = 7.50$$

*Can you use the expected utility approach to determine the optimal decision? Try Problem 1.*

Note that the optimal decision using the expected utility approach is $d_3$, do not invest. The ranking of alternatives according to the president's utility assignments and the associated monetary values is as follows:

| Ranking of Decision Alternatives | Expected Utility | Expected Monetary Value |
|---|---|---|
| Do not invest | 7.50 | \$ 0 |
| Investment A | 7.35 | \$9000 |
| Investment B | 6.55 | −\$1000 |

Note also that although investment A had the highest expected monetary value of \$9000, the analysis indicates that Swofford should decline this investment. The rationale behind not selecting investment A is that the 0.20 probability of a \$50,000 loss was considered to involve a serious risk by Swofford's president. The seriousness of this risk and its associated impact on the company were not adequately reflected by the expected monetary value of investment A. We assessed the utility for each payoff to assess this risk adequately.

## NOTES AND COMMENTS

1. In the Swofford problem we have been using a utility of 10 for the best payoff and 0 for the worst. We could have chosen any values so long as the utility associated with the best payoff exceeds the utility associated with the worst payoff. In practice, frequently a utility of 1 is associated with the best payoff and a utility of 0 is associated with the worst payoff. Had we made this choice, the utility for any monetary value $M$ would have been the value of $p$ at which the decision maker was indifferent between a guaranteed payoff of $M$ and a lottery in which the best payoff is obtained with probability $p$ and the worst payoff is obtained with probability $(1 - p)$. Thus, the utility for any monetary value would have been equal to the probability of earning the best payoff. Often this choice is made because of the ease in computation. We chose not to do so to emphasize the distinction between the utility values and the indifference probabilities for the lottery.
2. Other approaches to decision analysis, such as the optimistic, conservative, and minimax regret approaches, can be applied to utilities.

## Summary of Steps for Determining the Utility of Money

Before considering other aspects of utility, let us summarize the steps involved in determining the utility for a monetary value and using it within the decision analysis framework. The following steps state in general terms the procedure used to solve the Swofford, Inc., investment problem:

**Step 1.** Develop a payoff table using monetary values.

**Step 2.** Identify the best and worst payoff values in the table and assign each a utility value, with $U(\text{best payoff}) > U(\text{worst payoff})$.

**Step 3.** For every other monetary value $M$ in the original payoff table, do the following to determine its utility value:

a. Define the lottery: The best payoff is obtained with probability $p$ and the worst payoff is obtained with probability $(1 - p)$.

b. Determine the value of $p$ such that the decision maker is indifferent between a guaranteed payoff of $M$ and the lottery defined in step 3(a).

c. Calculate the utility of $M$ as follows:

$$U(M) = pU(\text{best payoff}) + (1 - p)U(\text{worst payoff})$$

**Step 4.** Convert the payoff table from monetary values to utility values.

**Step 5.** Apply the expected utility approach to the utility table developed in step 4 and select the decision alternative with the highest expected utility.

## NOTES AND COMMENTS

1. The procedure we described for determining the utility of monetary consequences can also be used to develop a utility measure for nonmonetary consequences. Assign the best consequence a utility of 10 and the worst a utility of 0. Then create a lottery with a probability of $p$ for the best consequence and $(1 - p)$ for the worst consequence. For each of the other consequences, find the value of $p$ that makes the decision maker indifferent between the lottery and the consequence. Then calculate the utility of the consequence in question as follows:

$$U(\text{consequence}) = pU(\text{best consequence}) + (1 - p)U(\text{worst consequence})$$

# 5.3 Utility: Other Considerations

In this section we describe how a risk-avoiding decision maker and a risk-taking decision maker differ in their assessment of utility. Expected utility is then used to show how a risk-avoiding decision maker and a risk-taking decision maker may prefer different decision alternatives for the same decision problem. We close this section by comparing expected monetary value and expected utility as criteria for decision making.

## Risk Avoiders Versus Risk Takers

The financial position of Swofford, Inc., was such that the firm's president evaluated investment opportunities from a conservative, or risk-avoiding, point of view. However, if the firm had a surplus of cash and a stable future, Swofford's president might have been looking for investment alternatives that, although perhaps risky, contained a potential for substantial profit. That type of behavior would demonstrate that the president is a risk taker with respect to this decision.

A **risk taker** is a decision maker who would choose a lottery over a guaranteed payoff when the expected value of the lottery is inferior to the guaranteed payoff. In this section we analyze the decision problem faced by Swofford from the point of view of a decision maker who would be classified as a risk taker. We then compare the conservative, or risk-avoiding, point of view of Swofford's president with the behavior of a decision maker who is a risk taker.

For the decision problem facing Swofford, Inc., using the general procedure for developing utilities as discussed in Section 5.2, a risk taker might express the utility for the various payoffs shown in Table 5.4. As before, $U(\$50{,}000) = 10$ and $U(-\$50{,}000) = 0$. Note the difference in behavior reflected in Table 5.4 and Table 5.2. That is, in determining the value of $p$ at which the decision maker is indifferent between a guaranteed payoff of $M$ and a lottery in which \$50,000 is obtained with probability $p$ and $-\$50{,}000$ with probability $(1 - p)$, the risk taker is willing to accept a greater risk of incurring a loss of \$50,000 in order to gain the opportunity to realize a profit of \$50,000.

To help develop the utility table for the risk taker, we have reproduced the Swofford, Inc., payoff table in Table 5.5. Using these payoffs and the risk taker's utility values given in Table 5.4, we can write the risk taker's utility table as shown in Table 5.6.

**TABLE 5.4** REVISED UTILITY VALUES FOR SWOFFORD, INC., ASSUMING A RISK TAKER

| Monetary Value | Indifference Value of *p* | Utility Value |
|---|---|---|
| \$50,000 | Does not apply | 10.0 |
| \$30,000 | 0.50 | 5.0 |
| \$20,000 | 0.40 | 4.0 |
| \$ 0 | 0.25 | 2.5 |
| −\$20,000 | 0.15 | 1.5 |
| −\$30,000 | 0.10 | 1.0 |
| −\$50,000 | Does not apply | 0.0 |



**TABLE 5.5** PAYOFF TABLE FOR SWOFFORD, INC.

| | State of Nature | | |
|---|---|---|---|
| **Decision Alternative** | **Prices Up $s_1$** | **Prices Stable $s_2$** | **Prices Down $s_3$** |
| **Investment A, $d_1$** | \$30,000 | \$20,000 | −\$50,000 |
| **Investment B, $d_2$** | \$50,000 | −\$20,000 | −\$30,000 |
| **Do not invest, $d_3$** | \$0 | \$0 | \$0 |



Using the state-of-nature probabilities $P(s_1) = 0.3$, $P(s_2) = 0.5$, and $P(s_3) = 0.2$, the expected utility for each decision alternative is

$$\text{EU}(d_1) = 0.3(5.0) + 0.5(4.0) + 0.2(0.0) = 3.50$$
$$\text{EU}(d_2) = 0.3(10.0) + 0.5(1.5) + 0.2(1.0) = 3.95$$
$$\text{EU}(d_3) = 0.3(2.5) + 0.5(2.5) + 0.2(2.5) = 2.50$$

What is the recommended decision? Perhaps somewhat to your surprise, the analysis recommends investment B, with the highest expected utility of 3.95. Recall that this investment has a −\$1000 expected monetary value. Why is it now the recommended decision? Remember that the decision maker in this revised problem is a risk taker. Thus, although the expected value of investment B is negative, utility analysis has shown that this decision maker is enough of a risk taker to prefer investment B and its potential for the \$50,000 profit.

**TABLE 5.6** UTILITY TABLE OF A RISK TAKER FOR SWOFFORD, INC.

| | State of Nature | | |
|---|---|---|---|
| **Decision Alternative** | **Prices Up $s_1$** | **Prices Stable $s_2$** | **Prices Down $s_3$** |
| **Investment A, $d_1$** | 5.0 | 4.0 | 0.0 |
| **Investment B, $d_2$** | 10.0 | 1.5 | 1.0 |
| **Do not invest, $d_3$** | 2.5 | 2.5 | 2.5 |

The expected utility values give the following order of preference of the decision alternatives for the risk taker and the associated expected monetary values:

| Ranking of Decision Alternatives | Expected Utility | Expected Monetary Value |
|---|---|---|
| Investment B | 3.95 | −$1000 |
| Investment A | 3.50 | $9000 |
| Do not invest | 2.50 | $0 |

Comparing the utility analysis for a risk taker with the more conservative preferences of the president of Swofford, Inc., who is a risk avoider, we see that, even with the same decision problem, different attitudes toward risk can lead to different recommended decisions. The utility values established by Swofford's president indicated that the firm should not invest at this time, whereas the utilities established by the risk taker showed a preference for investment B. Note that both of these decisions differ from the best expected monetary value decision, which was investment A.

We can obtain another perspective of the difference between behaviors of a risk avoider and a risk taker by developing a graph that depicts the relationship between monetary value and utility. We use the horizontal axis of the graph to represent monetary values and the vertical axis to represent the utility associated with each monetary value. Now, consider the data in Table 5.2, with a utility value corresponding to each monetary value for the original Swofford, Inc., problem. These values can be plotted on a graph such as that in Figure 5.1, and a curve can be drawn through the observed points. The resulting curve is the **utility**

**FIGURE 5.1** UTILITY FUNCTION FOR MONEY FOR THE RISK AVOIDER

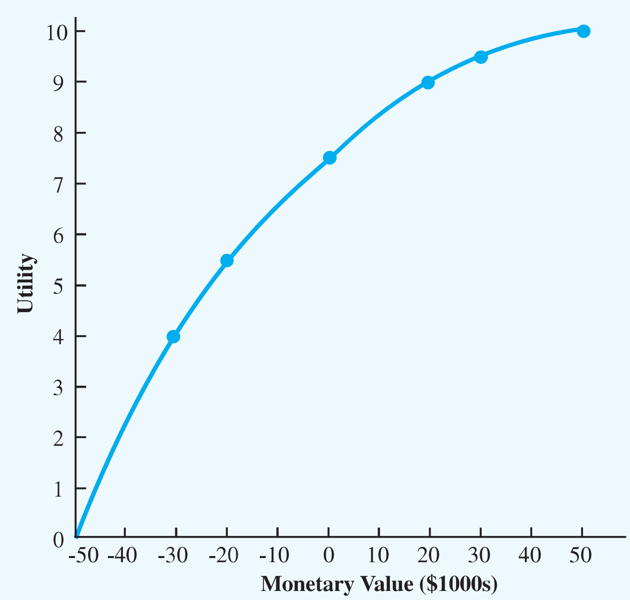
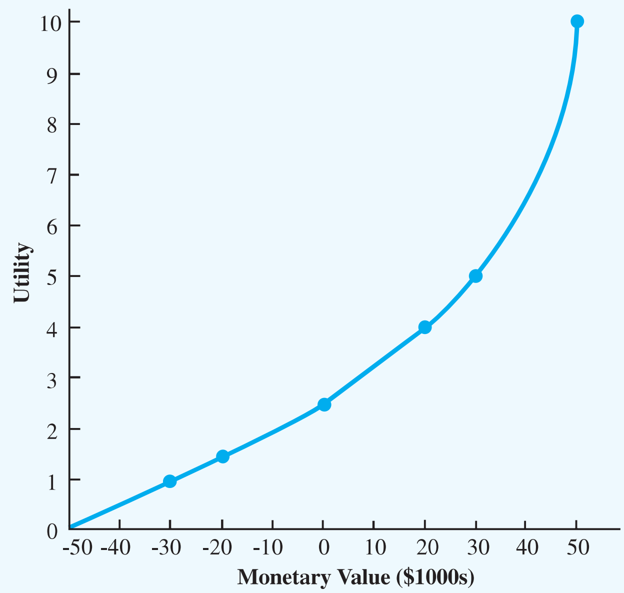


FIGURE 5.2 UTILITY FUNCTION FOR MONEY FOR THE RISK TAKER






**function for money** for Swofford's president. Recall that these points reflected the conservative, or risk-avoiding, nature of Swofford's president. Hence, we refer to the curve in Figure 5.1 as a utility function for a risk avoider. Using the data in Table 5.4 developed for a risk taker, we can plot these points on a graph such as that in Figure 5.2. The resulting curve depicts the utility function for a risk taker.

By looking at the utility functions of Figures 5.1 and 5.2, we can begin to generalize about the utility functions for risk avoiders and risk takers. Although the exact shape of the utility function will vary from one decision maker to another, we can see the general shape of these two types of utility functions. The utility function for a risk avoider shows a diminishing marginal return for money. For example, the increase in utility going from a monetary value of −$30,000 to $0 is $7.5 - 4.0 = 3.5$, whereas the increase in utility in going from $0 to $30,000 is only $9.5 - 7.5 = 2.0$.

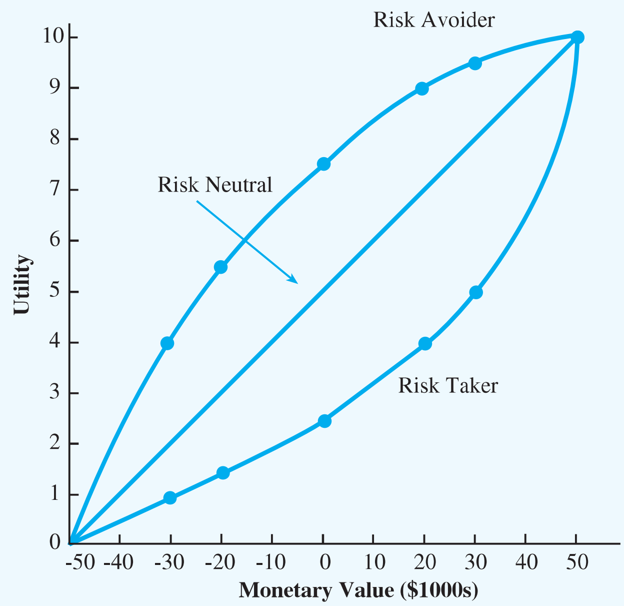
However, the utility function for a risk taker shows an increasing marginal return for money. For example, in Figure 5.2, the increase in utility in going from −$30,000 to $0 is $2.5 - 1.0 = 1.5$, whereas the increase in utility in going from $0 to $30,000 is $5.0 - 2.5 = 2.5$. Note also that in either case the utility function is always increasing; that is, more money leads to more utility. All utility functions possess this property.

*Try Problem 5 for practice in plotting the utility function for risk-avoider, risk-taker, and risk-neutral decision makers.*

We have concluded that the utility function for a risk avoider shows a diminishing marginal return for money and that the utility function for a risk taker shows an increasing marginal return. When the marginal return for money is neither decreasing nor increasing but remains constant, the corresponding utility function describes the behavior of a decision maker who is neutral to risk. The following characteristics are associated with a **risk-neutral decision maker**:

1. The utility function can be drawn as a straight line connecting the "best" and the "worst" points.

FIGURE 5.3 UTILITY FUNCTION FOR RISK-AVOIDER, RISK-TAKER, AND RISK-NEUTRAL DECISION MAKERS






2. The expected utility approach and the expected value approach applied to monetary payoffs result in the same action.

Figure 5.3 depicts the utility function of a risk-neutral decision maker using the Swofford, Inc., problem data. For comparison purposes, we also show the utility functions for the cases where the decision maker is either a risk taker or a risk avoider.

Generally, when the payoffs for a particular decision-making problem fall into a reasonable range—the best is not too good and the worst is not too bad—decision makers tend to express preferences in agreement with the expected monetary value approach. Thus, we suggest asking the decision maker to consider the best and worst possible payoffs for a problem and assess their reasonableness. If the decision maker believes that they are in the reasonable range, the decision alternative with the best expected monetary value can be used. However, if the payoffs appear unreasonably large or unreasonably small (for example, a huge loss) and if the decision maker believes that monetary values do not adequately reflect her or his true preferences for the payoffs, a utility analysis of the problem should be considered.

Unfortunately, determination of the appropriate utilities is not a trivial task. As we have shown, measuring utility requires a degree of subjectivity on the part of the decision maker, and different decision makers will have different utility functions. This aspect of utility often causes decision makers to feel uncomfortable about using the expected utility approach. However, if you encounter a decision situation in which you are convinced that monetary value is not the sole measure of performance, and if you agree that a quantitative analysis of the decision problem is desirable, you should recommend that utility analysis be considered.

## NOTES AND COMMENTS

1. Circumstances often dictate whether one acts as a risk avoider or a risk taker when making a decision. You may think of yourself as a risk avoider when faced with financial decisions, but if you have ever purchased a lottery ticket you have actually acted as a risk taker. For example, suppose you purchase a $1.00 lottery ticket for a simple lottery in which the object is to pick the six numbers that will be drawn from 50 potential numbers. Also suppose that if you win (correctly choose all six numbers that are drawn), you will receive $1,000,000. There are 15,890,700 possible winning combinations, so your probability of winning is $1/15{,}890{,}700 \approx 0.00000006293$ (i.e., *very low*) and the expected monetary value of your ticket is

$$\begin{aligned}&\frac{1}{15{,}890{,}700}(\$1{,}000{,}000 - 1\$)\\&\quad + \left(1 - \frac{1}{15{,}890{,}700}\right)(-\$1)\\&= \frac{1}{15{,}890{,}700}(\$1{,}000{,}000 - \$1)\\&\quad + \frac{15{,}890{,}699}{15{,}890{,}700}(-\$1)\\&\approx -\$0.94\end{aligned}$$

(Why is the payoff for winning equal to $1,000,000 − $1? You had to pay for your ticket!)

If a lottery ticket has a negative expected value, why does anyone play? The answer is in utility—most people who play lotteries associate great utility with the possiblity of winning the $1,000,000 prize and relatively little utility with the $1.00 cost for a ticket, and so the expected value of the utility of the lottery ticket is positive even though the expected monetary value of the ticket is negative.

2. In many decision-making problems, expected monetary value and expected utility will lead to identical recommendations. In fact, this result will always be true if the decision maker is risk neutral. In general, if the decision maker is almost risk neutral over the range of payoffs (from lowest to highest) for a particular decision problem, the decision alternative with the best expected monetary value leads to selection of the most preferred decision alternative. The trick lies in recognizing the range of monetary values over which a decision maker's utility function is risk neutral.

# 5.4 Introduction to Game Theory

In decision analysis, a single decision maker seeks to select an optimal decision alternative after considering the possible outcomes of one or more chance events. In **game theory**, two or more decision makers are called players, and they compete as adversaries against each other. Each player selects a strategy independently without knowing in advance the strategy of the other player or players. The combination of the competing strategies provides the value of the game to the players. Game theory applications have been developed for situations in which the competing players are teams, companies, political candidates, armies, and contract bidders.

*Until 1944, when Von Neumann and Morgenstern published the book* Theory of Games and Economic Behavior, *the literature on decisions involving risk consisted primarily of applications involving the use of probability in gambling.*

In this section we describe **two-person, zero-sum games**. *Two-person* means that two competing players take part in the game. *Zero-sum* means that the gain (or loss) for one player is equal to the corresponding loss (or gain) for the other player (what one player wins, the other player loses). As a result, the gain and loss balance out so that the game results in the sum of zero. Let us demonstrate a two-person, zero-sum game and its solution by considering two companies competing for market share.

## Competing for Market Share

Suppose that two companies are the only manufacturers of a particular product; they compete against each other for market share. In planning a marketing strategy for the coming year, each company is considering three strategies designed to take market share from the other company. The three strategies, assumed to be the same for both companies, are as follows:

**Strategy 1** Increase advertising

**Strategy 2** Provide quantity discounts

**Strategy 3** Extend product warranty

A payoff table showing the percentage gain in the market share for Company A expected for each combination of strategies follows. The notations $a_1$, $a_2$, and $a_3$ identify the three strategies for Company A; the notations $b_1$, $b_2$, and $b_3$ identify the three strategies for Company B. It is a zero-sum game because any gain in market share for Company A is a loss in market share for Company B.

| | | Company B | | |
|---|---|---|---|---|
| | | **Increase Advertising $b_1$** | **Quantity Discounts $b_2$** | **Extend Warranty $b_3$** |
| Company A | **Increase Advertising, $a_1$** | 4 | 3 | 2 |
| | **Quantity Discount $a_2$** | −1 | 4 | 1 |
| | **Extend Warranty, $a_3$** | 5 | −2 | 0 |

In interpreting the entries in the table we see that if Company A increases advertising ($a_1$) and Company B increases advertising ($b_1$), Company A will come out ahead with an increase in market share of 4%; because this is a zero-sum game, Company B will suffer a corresponding 4% decrease in market share. On the other hand, if Company A provides quantity discounts ($a_2$) and Company B increases advertising ($b_1$), Company A is projected to lose 1% of market share to Company B. Company A is seeking payoff values that show relatively large increases in its market share. Company B is seeking payoff values that show decreases or small increases in Company A's market share, and thus better results for Company B.

This game involving market share meets the requirements of a two-person, zero-sum game. The two companies are the two players and the zero-sum occurs because the gain (or loss) in market share for Company A is the same as the loss (or gain) in market share for Company B. Due to the planning horizon, each company must select a strategy before knowing the other company's strategy. What are the optimal strategies for the two companies?

The logic of game theory assumes that each company or player has the same information and will select a strategy that provides the best possible outcome from its point of view. Suppose Company A selects strategy $a_1$. Market share increases of 4%, 3%, or

2% are possible depending upon Company B's strategy. If Company B believes that Company A will use strategy $a_1$, then Company B will employ strategy $b_3$. Under the assumption that Company B will select the strategy that is best for it, Company A analyzes the game by protecting itself against the actions of Company B. In doing so, Company A identifies the minimum possible payoff for each of its actions. This payoff is the minimum value in each row of the payoff matrix. These row minimums are computed in the payoff table as follows:

| | | Company B | | | |
|---|---|---|---|---|---|
| | | **Increase Advertising $b_1$** | **Quantity Discounts $b_2$** | **Extend Warranty $b_3$** | **Minimum** |
| Company A | **Increase Advertising, $a_1$** | 4 | 3 | 2 | (2) |
| | **Quantity Discounts, $a_2$** | −1 | 4 | 1 | −1 |
| | **Extend Warranty, $a_3$** | 5 | −2 | 0 | −2 |

Maximum of row minimums

Considering the entries in the Minimum column, we see that Company A can be guaranteed an increase in market share of at least 2% by selecting the strategy that provides the *maximum of the row minimums* (strategy $a_1$). Thus, Company A follows a *maximin* procedure and selects strategy $a_1$ as its best strategy.

Let us now look at the payoff table from the point of view of the other player, Company B. The entries in the payoff table represent losses in market share. Consider what happens to Company B if strategy $b_1$ is selected. Market share decreases of 4%, –1%, and 5% are possible. Under the assumption that Company A will select the strategy that is best for it, Company B knows that if it selects strategy $b_1$, a loss in market share of as much as 5% could be incurred. Thus, Company B analyzes the game by considering the maximum value in each column, which provides the maximum decrease in its market share for each Company A strategy. These column maximums are computed as follows:

| | | Company B | | | |
|---|---|---|---|---|---|
| | | **Increase Advertising $b_1$** | **Quantity Discounts $b_2$** | **Extend Warranty $b_3$** | **Minimum** |
| Company A | **Increase Advertising, $a_1$** | 4 | 3 | 2 | (2) |
| | **Quantity Discounts $a_2$** | −1 | 4 | 1 | −1 |
| | **Extend Warranty, $a_3$** | 5 | −2 | 0 | −2 |
| | **Maximum** | 5 | 4 | (2) | |

Minimum of column maximums

Maximum of row minimums

By considering the entries in the Maximum row, Company B can be guaranteed a decrease in market share of no more than 2% by selecting the strategy that provides the *minimum of the column maximums* (strategy $b_3$). Thus, Company B follows a *minimax* procedure and selects strategy $b_3$ as its best strategy. Under strategy $b_3$, Company B knows that Company A cannot gain more than 2% in market share.

## Identifying a Pure Strategy

Whenever the maximum of the row minimums *equals* the minimum of the column maximums, the players cannot improve their outcomes by changing strategies. The game is said to have a **saddle point**. With a saddle point, the optimal strategies and the value of the game cannot be improved by either player changing strategies. Thus, a **pure strategy** has been identified as being optimal for both players. The requirement for a pure strategy is as follows:

$$\text{Maximum(Row minimums)} = \text{Minimum(Column maximums)}$$

That is, the maximin value for Player A equals the minimax value for Player B, and so Player A would not alter his strategy even if he knew Player B's strategy and Player B would not alter his strategy even if he knew Player A's strategy. In our example, the solution to the game is a pure strategy. If Company A increases its advertising (strategy $a_1$), Company B's optimal strategy is to extend its product warranty (strategy $b_3$); if Company B chooses to extend its product warranty (strategy $b_3$), Company A's optimal strategy is to increase its advertising (strategy $a_1$). The value of the game shows that this optimal solution will increase Company A's market share by 2% and decrease Company B's market share by 2%.

With a pure strategy, neither player can improve its position by changing to a different strategy. In our marketing example, the pure strategy for Company A is $a_1$. When Company B selects its pure strategy $b_3$, the value of the game shows an increase in Company A's market share of 2%. Note that if Company B tries to change its pure strategy from $b_3$, Company A's market share will increase 4% if $b_1$ is selected or will increase 3% if $b_2$ is selected. Company B must stay with its pure strategy $b_3$ to obtain its best result. Similarly, note that if Company A tries to change its pure strategy from $a_1$, Company A's market share will increase only 1% if $a_2$ is selected or will not increase at all if $a_3$ is selected. Company A must stay with its pure strategy $a_1$ in order to keep its 2% increase in market share. Thus, even if one of the players discovered in advance the opponent's strategy, no advantage could be gained by switching to a different strategy.

*If a pure strategy exists, it is the optimal solution for the game.*

When a pure strategy is optimal for a two-person, zero-sum game, the following steps will find the optimal strategy for each player:

**Step 1.** Compute the minimum payoff for each row (Player A).
**Step 2.** For Player A, select the strategy that provides the *maximum* of the row minimums.
**Step 3.** Compute the maximum payoff for each column (Player B).
**Step 4.** For Player B, select the strategy that provides the *minimum* of the column maximums.
**Step 5.** If the maximin value (step 2) equals the minimax value (step 4), an optimal pure strategy exists for both players. The optimal strategy for Player A is identified in step 2, and the optimal strategy for Player B is identified in step 4. The value of the game is given by the value at the saddle point where the optimal strategies for both players intersect.

If in step 5 the maximin value for Player A does not equal the minimax value for Player B, a pure strategy is not optimal for the two-person, zero-sum game. In this case, a *mixed strategy* is best. In the next section we show when it is necessary to employ a mixed strategy.

## 5.5 Mixed Strategy Games

Consider the two-person, zero-sum game that occurs in a football game. The two competing players are the two football teams. On each play, the game is zero-sum because the yardage gained by one team is equal to the yardage lost by the other team. As usual in game theory, each team must select its strategy before knowing the strategy selected by the other team. In this example, let Team A be the team on offense trying to gain yardage and Team B be the team on defense trying to keep the yardage gained by Team A to a minimum. We define the offensive strategies for Team A as follows:

$$a_1 = \text{running play}$$
$$a_2 = \text{passing play}$$

The defensive strategies for Team B are as follows:

$$b_1 = \text{run defense}$$
$$b_2 = \text{pass defense}$$

The payoff table shows the yardage gained by Team A depending upon the strategies selected by the two teams.

| | | Team B | |
|---|---|---|---|
| | | **Run Defense** $b_1$ | **Pass Defense** $b_2$ |
| **Team A** | **Run,** $a_1$ | 1 | 6 |
| | **Pass,** $a_2$ | 15 | 0 |

Applying the five-step procedure used to identify a pure strategy, the row minimums and the column maximums are as follows:

| | | Team B | | |
|---|---|---|---|---|
| | | **Run Defense** $b_1$ | **Pass Defense** $b_2$ | **Minimum** |
| **Team A** | **Run,** $a_1$ | 1 | 6 | (1) |
| | **Pass,** $a_2$ | 15 | 0 | 0 |
| | **Maximum** | 15 | (6) | |

The maximum of the row minimums is 1 and the minimum of the column maximums is 6. Because these values are not equal, the two-person, zero-sum game does not have an optimal pure strategy. In this case, a **mixed strategy** solution is best. With a mixed strategy the optimal solution for each player is to randomly select among the alternative strategies. In the football example, then, the offensive Team A will mix up or vary its selection of running ($a_1$) and passing ($a_2$) plays, while the defensive Team B will mix up or vary its selection of a run defense ($b_1$) and a pass defense ($b_2$).

When you think about a football game, it becomes clear that a pure strategy such as Team A always selecting a running play would not work. Team B would recognize Team

A's pure strategy and would always be prepared with a run defense. Thus, a Team A mixed strategy of sometimes running and sometimes passing would make sense. When a mixed strategy solution is needed, game theory will determine the optimal probabilities for each strategy for each player. That is, the game theory solution of the football example will tell the offensive team the optimal probabilities for a running play and a passing play. At the same time, the solution will tell the defensive team the optimal probabilities for a run defense and a pass defense. The following discussion shows how to calculate these mixed strategy probabilities.

Let

$$p = \text{the probability Team A selects a running play}$$
$$(1 - p) = \text{the probability Team A selects a passing play}$$

When a mixed strategy solution exists, we seek to determine the probability $p$ for Team A such that Team B cannot improve its result by changing its defensive strategy. First assume that Team B selects a run defense as shown in column $b_1$. If Team A selects a running play with probability $p$ and a passing play with probability $(1 - p)$, the expected value of the yardage gain for Team A is computed as follows:

If Team B selects $b_1$:

$$\text{EV(Yardage)} = 1p + 15(1 - p)$$

If Team B selects its pass defense as shown in column $b_2$, the expected value of the yardage gain for Team A will be as follows:

If Team B selects $b_2$:

$$\text{EV(Yardage)} = 6p + 0(1 - p) = 6p$$

To guarantee that Team B cannot change its strategy and decrease the expected value of the yardage gained by Team A, we set the two expected values equal and solve for the value of $p$.

$$1p + 15(1 - p) = 6p$$
$$1p + 15 - 15p = 6p$$
$$20p = 15$$
$$p = 15/20 = 0.75$$

With $p = 0.75$, $(1 - p) = 1 - 0.75 = 0.25$. This result tells Team A it should select a running play with a 0.75 probability and a passing play with a 0.25 probability. The expected value of the yardage gained, which is the *value of the game,* is

$$\text{EV(Yardage)} = 1p + 15(1 - p) = 1(0.75) + 15(0.25) = 4.5 \text{ yards per play}$$

Now let us consider the optimal probabilities for Team B. Let

$$q = \text{the probability Team B selects a run defense}$$
$$(1 - q) = \text{the probability Team B selects a pass defense}$$

Using the same logic we used for computing Team A's optimal probabilities, we want to determine the value of $q$ such that Team A cannot increase the expected value of the yardage gained by changing its offensive strategy. We first compute the expected value of the yardage for Team B for the following two cases:

If Team A selects $a_1$:

$$\text{EV(Yardage)} = 1q + 6(1 - q)$$

If Team A selects $a_2$:

$$\text{EV(Yardage)} = 15q + 0(1 - q) = 15q$$

To guarantee that Team A cannot change its strategy and affect the expected value of the yardage for Team B, we set the two expected values equal and solve for the value of $q$ as follows:

$$\begin{aligned} 1q + 6(1 - q) &= 15q \\ 1q + 6 - 6q &= 15q \\ 20q &= 6 \\ q &= 6/20 = 0.30 \end{aligned}$$

With $q = 0.30$, $(1 - q) = 1 - 0.30 = 0.70$. This result tells Team B that it should select a run defense with a 0.30 probability and a pass defense with a 0.70 probability. The expected yardage gained, which is the value of the game, will remain 4.5 yards per play.

Thus, we have the optimal mixed strategy solution for the football game example. Any $2 \times 2$ two-person, zero-sum mixed strategy game can be solved algebraically as shown in this example. If a larger two-person, zero-sum game involves a mixed strategy, solving it is a bit more complicated.

## A Larger Mixed Strategy Game

Consider the following two-person, zero-sum game:

| | | Player B | | |
|---|---|---|---|---|
| | | $b_1$ | $b_2$ | $b_3$ |
| Player A | $a_1$ | 0 | −1 | 2 |
| | $a_2$ | 5 | 4 | −3 |
| | $a_3$ | 2 | 3 | −4 |

Following the usual procedure for identifying a pure strategy, we compute the row minimums and the column maximums:

| | | Player B | | | |
|---|---|---|---|---|---|
| | | $b_1$ | $b_2$ | $b_3$ | Minimum |
| Player A | $a_1$ | 0 | −1 | 2 | (−1) |
| | $a_2$ | 5 | 4 | −3 | −3 |
| | $a_3$ | 2 | 3 | −4 | −4 |
| | Maximum | 5 | 4 | (2) | |

The maximum of the row minimums is –1 and the minimum of the column maximums is 2. Because the maximin and minimax values are not equal, the two-person, zero-sum game does not have an optimal pure strategy. However, with a problem larger than 2 × 2, we cannot use the algebraic solution for the mixed strategy probabilities as we did in the previous example.

If a game larger than 2 × 2 requires a mixed strategy, we first look for dominated strategies in order to reduce the size of the game. A **dominated strategy** exists if another strategy *is at least as good* regardless of what the opponent does. For example, consider strategies $a_2$ and $a_3$. The payoff table shows that in column $b_1$, $5 > 2$; in column $b_2$, $4 > 3$; and in column $b_3$, $-3 > -4$. Thus, regardless of what Player B does, Player A will always prefer the higher values of strategy $a_2$ compared to strategy $a_3$. Thus, strategy $a_3$ is dominated by strategy $a_2$ and can be dropped from consideration by Player A. Eliminating dominated strategies from the game reduces its size. After eliminating $a_3$, the reduced game becomes

| | | Player B | | |
|---|---|---|---|---|
| | | $b_1$ | $b_2$ | $b_3$ |
| Player A | $a_1$ | 0 | −1 | 2 |
| | $a_2$ | 5 | 4 | −3 |

Next, we look for more dominated strategies. Player A finds no other dominated strategies. However, consider strategies $b_1$ and $b_2$ for Player B. Remember that Player B is interested in smaller values. The payoff table shows that in row $a_1$, $-1 < 0$, and in row $a_2$, $4 < 5$. Thus, regardless of what Player A does, Player B would always prefer the smaller values of strategy $b_2$ compared to strategy $b_1$. Thus, strategy $b_1$ is dominated by strategy $b_2$ and can be eliminated from the game. With this dominated strategy eliminated, the reduced game becomes

| | | Player B | |
|---|---|---|---|
| | | $b_1$ | $b_3$ |
| Player A | $a_1$ | −1 | 2 |
| | $a_2$ | 4 | −3 |

*Problem 14 at the end of the chapter will ask you to find the optimal probabilities for this example.*

By successively eliminating dominated strategies, we reduce the game to a 2 × 2 game. The algebraic solution procedure described earlier in this section can now be used to identify the optimal probabilities for the mixed strategy solution.

*Identifying and eliminating dominated strategies may reduce the game to a 2 × 2 game. If so, an algebraic procedure may be used to determine the mixed strategy solution.*

Finally, it is important to realize that no hard-and-fast rule identifies dominated strategies. Basically, the analyst must make pairwise comparisons of the decision strategies in an attempt to identify dominated strategies. The goal is to identify and eliminate dominated strategies sequentially in order to reduce the game to a 2 × 2 game so that an algebraic solution procedure can be used to solve for the mixed strategy probabilities.

## Summary of Steps for Solving Two-Person, Zero-Sum Games

The following summary shows the steps used to solve two-person, zero-sum games:

1. Use the maximin procedure for Player A and the minimax procedure for Player B to determine whether a pure strategy solution exists. (See previous steps for identifying a pure strategy.) If a pure strategy exists, it is the optimal solution.
2. If a pure strategy does not exist and the game is larger than $2 \times 2$, identify a dominated strategy to eliminate a row or column. Develop the reduced payoff table and continue to use dominance to eliminate as many additional rows and columns as possible.
3. If the reduced game is $2 \times 2$, solve for the optimal mixed strategy probabilities algebraically.

*In 1994, John Harsanui. John Nash, and Reinhard Selten received the Nobel Prize in Economics for their work on noncooperative game theory.*

If the game cannot be reduced to a $2 \times 2$ game, a linear programming model can be used to solve for the optimal mixed strategy probabilities. The formulation of a linear programming model to solve these larger game theory problems is beyond the scope of this text.

## Extensions

*Problem 17 at the end of the chapter demonstrates a simple constant-sum game theory problem.*

We presented the basic model for two-person, zero-sum games. However, game theory models extend beyond two-person, zero-sum games. One extension is a two-person, constant-sum game that occurs when the payoffs for the strategies chosen sum to a constant other than zero. In addition, game theory can be extended to include more general $n$-person games. Cooperative games where players are allowed preplay communications are another variation. Finally, some game theory models allow an infinite number of strategies to be available for the players.

## Summary

In this chapter we showed how utility could be used in decision-making situations in which monetary value did not provide an adequate measure of the payoffs. Utility is a measure of the total worth of a consequence. As such, utility takes into account the decision maker's assessment of all aspects of a consequence, including profit, loss, risk, and perhaps additional nonmonetary factors. The examples showed how the use of expected utility can lead to decision recommendations that differ from those based on expected monetary value.

A decision maker's judgment must be used to establish the utility for each consequence. We presented a step-by-step procedure to determine a decision maker's utility for monetary payoffs. We also discussed how conservative, risk-avoiding decision makers assess utility differently from more aggressive, risk-taking decision makers. If the decision maker is risk neutral, we showed that the solution using expected utility is identical to the solution using expected monetary value.

We presented an introduction to game theory by describing how to solve two-person, zero-sum games. In these games, the two players end up with the sum of the gain (loss) to one player and the loss (gain) to the other player always equal to zero. We described the steps that can be used to determine whether a two-person, zero-sum game results in an optimal pure strategy. If a pure strategy is optimal, a saddle point determines the value of the game. If an optimal pure strategy does not exist for a two-person, zero-sum $2 \times 2$ game, we showed how to identify an optimal mixed strategy. With a mixed strategy, each player uses probability to select a strategy for each play of the game. We showed how dominance could be used to reduce the size of mixed strategy games. If the elimination of dominated strategies can reduce a larger game to a 2 × 2 game, an algebraic solution procedure can be used to find a solution. If the game cannot be reduced to a 2 × 2 game, a linear programming model is needed to determine the optimal mixed strategy solution.

## Glossary

**Utility** A measure of the total worth of a consequence reflecting a decision maker's attitude toward considerations such as profit, loss, and risk.
**Lottery** A hypothetical investment alternative with a probability $p$ of obtaining the best payoff and a probability of $(1 - p)$ of obtaining the worst payoff.
**Risk avoider** A decision maker who would choose a guaranteed payoff over a lottery with a better expected payoff.
**Expected utility (EU)** The weighted average of the utilities associated with a decision alternative. The weights are the state-of-nature probabilities.
**Risk taker** A decision maker who would choose a lottery over a better guaranteed payoff.
**Utility function for money** A curve that depicts the relationship between monetary value and utility.
**Risk-neutral decision maker** A decision maker who is neutral to risk. For this decision maker the decision alternative with the best expected monetary value is identical to the alternative with the highest expected utility.
**Game theory** The study of decision situations in which two or more players compete as adversaries. The combination of strategies chosen by the players determines the value of the game to each player.
**Two-person, zero-sum game** A game with two players in which the gain to one player is equal to the loss to the other player.
**Saddle point** A condition that exists when pure strategies are optimal for both players in a two-person, zero-sum game. The saddle point occurs at the intersection of the optimal strategies for the players, and the value of the saddle point is the value of the game.
**Pure strategy** A game solution that provides a single best strategy for each player.
**Mixed strategy** A game solution in which the player randomly selects the strategy to play from among several strategies with positive probabilities. The solution to the mixed strategy game identifies the probabilities that each player should use to randomly select the strategy to play.
**Dominated strategy** A strategy is dominated if another strategy is at least as good for every strategy that the opposing player may employ. A dominated strategy will never be selected by the player and as such can be eliminated in order to reduce the size of the game.

## Problems

SELF test

1. A firm has three investment alternatives. Payoffs are in thousands of dollars.

| | Economic Conditions | | |
|---|---|---|---|
| **Decision Alternative** | **Up $s_1$** | **Stable $s_2$** | **Down $s_3$** |
| Investment A, $d_1$ | 100 | 25 | 0 |
| Investment B, $d_2$ | 75 | 50 | 25 |
| Investment C, $d_3$ | 50 | 50 | 50 |
| Probabilities | 0.40 | 0.30 | 0.30 |

a. Using the expected value approach, which decision is preferred?
b. For the lottery having a payoff of \$100,000 with probability $p$ and \$0 with probability $(1 - p)$, two decision makers expressed the following indifference probabilities. Find the most preferred decision for each decision maker using the expected utility approach.

| | Indifference Probability ($p$) | |
|---|---|---|
| **Profit** | **Decision Maker A** | **Decision Maker B** |
| \$75,000 | 0.80 | 0.60 |
| \$50,000 | 0.60 | 0.30 |
| \$25,000 | 0.30 | 0.15 |

c. Why don't decision makers A and B select the same decision alternative?

2. Alexander Industries is considering purchasing an insurance policy for its new office building in St. Louis, Missouri. The policy has an annual cost of \$10,000. If Alexander Industries doesn't purchase the insurance and minor fire damage occurs, a cost of \$100,000 is anticipated; the cost if major or total destruction occurs is \$200,000. The costs, including the state-of-nature probabilities, are as follows:

| | | Damage | |
|---|---|---|---|
| **Decision Alternative** | **None $s_1$** | **Minor $s_2$** | **Major $s_3$** |
| Purchase insurance, $d_1$ | 10,000 | 10,000 | 10,000 |
| Do not purchase insurance, $d_2$ | 0 | 100,000 | 200,000 |
| Probabilities | 0.96 | 0.03 | 0.01 |

a. Using the expected value approach, what decision do you recommend?
b. What lottery would you use to assess utilities? (*Note:* Because the data are costs, the best payoff is \$0.)
c. Assume that you found the following indifference probabilities for the lottery defined in part (b). What decision would you recommend?

| Cost | Indifference Probability |
|---|---|
| 10,000 | $p = 0.99$ |
| 100,000 | $p = 0.60$ |

d. Do you favor using expected value or expected utility for this decision problem? Why?

3. In a certain state lottery, a lottery ticket costs \$2. In terms of the decision to purchase or not to purchase a lottery ticket, suppose that the following payoff table applies:

| | State of Nature | |
|---|---|---|
| **Decision Alternatives** | **Win $s_1$** | **Loses $s_2$** |
| Purchase lottery ticket, $d_1$ | 300,000 | −2 |
| Do not purchase lottery ticket, $d_2$ | 0 | 0 |

a. A realistic estimate of the chances of winning is 1 in 250,000. Use the expected value approach to recommend a decision.
b. If a particular decision maker assigns an indifference probability of 0.000001 to the \$0 payoff, would this individual purchase a lottery ticket? Use expected utility to justify your answer.

4. Two different routes accommodate travel between two cities. Route A normally takes 60 minutes and route B normally takes 45 minutes. If traffic problems are encountered on route A, the travel time increases to 70 minutes; traffic problems on route B increase travel time to 90 minutes. The probability of a delay is 0.20 for route A and 0.30 for route B.
   a. Using the expected value approach, what is the recommended route?
   b. If utilities are to be assigned to the travel times, what is the appropriate lottery? (*Note:* The smaller times should reflect higher utilities.)
   c. Use the lottery of part (b) and assume that the decision maker expresses indifference probabilities of

$$p = 0.80 \quad \text{for 60 minutes}$$
$$p = 0.60 \quad \text{for 70 minutes}$$

   What route should this decision maker select? Is the decision maker a risk taker or a risk avoider?

SELF test

5. Three decision makers have assessed utilities for the following decision problem (payoff in dollars):

| Decision Alternative | State of Nature $s_1$ | $s_2$ | $s_3$ |
|---|---|---|---|
| $d_1$ | 20 | 50 | −20 |
| $d_2$ | 80 | 100 | −100 |

The indifference probabilities are as follows:

| Payoff | Indifference Probability ($p$) Decision Maker A | Decision Maker B | Decision Maker C |
|---|---|---|---|
| 100 | 1.00 | 1.00 | 1.00 |
| 80 | 0.95 | 0.70 | 0.90 |
| 50 | 0.90 | 0.60 | 0.75 |
| 20 | 0.70 | 0.45 | 0.60 |
| −20 | 0.50 | 0.25 | 0.40 |
| −100 | 0.00 | 0.00 | 0.00 |

   a. Plot the utility function for money for each decision maker.
   b. Classify each decision maker as a risk avoider, a risk taker, or risk neutral.
   c. For the payoff of 20, what is the premium that the risk avoider will pay to avoid risk? What is the premium that the risk taker will pay to have the opportunity of the high payoff?

6. In Problem 5, if $P(s_1) = 0.25$, $P(s_2) = 0.50$, and $P(s_3) = 0.25$, find a recommended decision for each of the three decision makers. (*Note:* For the same decision problem, different utilities can lead to different decisions.)

7. Suppose that the point spread for a particular sporting event is 10 points and that with this spread you are convinced you would have a 0.60 probability of winning a bet on your team. However, the local bookie will accept only a $1000 bet. Assuming that such bets are legal, would you bet on your team? (Disregard any commission charged by the bookie.) Remember that *you* must pay losses out of your own pocket. Your payoff table is as follows:

| | State of Nature | |
|---|---|---|
| **Decision Alternatives** | **You Win** | **You Lose** |
| Bet | $1000 | −$1000 |
| Don't bet | $0 | $0 |

   a. What decision does the expected value approach recommend?
   b. What is *your* indifference probability for the $0 payoff? (Although this choice isn't easy, be as realistic as possible. It is required for an analysis that reflects your attitude toward risk.)
   c. What decision would you make based on the expected utility approach? In this case are you a risk taker or a risk avoider?
   d. Would other individuals assess the same utility values you do? Explain.
   e. If your decision in part (c) was to place the bet, repeat the analysis assuming a minimum bet of $10,000.

8. A Las Vegas roulette wheel has 38 different numerical values. If an individual bets on one number and wins, the payoff is 35 to 1.
   a. Show a payoff table for a $10 bet on one number for decision alternatives of bet and do not bet.
   b. What is the recommended decision using the expected value approach?
   c. Do the Las Vegas casinos want risk-taking or risk-avoiding customers? Explain.
   d. What range of utility values would a decision maker have to assign to the $0 payoff in order to have expected utility justify a decision to place the $10 bet?

9. A new product has the following profit projections and associated probabilities:

| **Profit** | **Probability** |
|---|---|
| $150,000 | 0.10 |
| $100,000 | 0.25 |
| $ 50,000 | 0.20 |
| $0 | 0.15 |
| −$ 50,000 | 0.20 |
| −$100,000 | 0.10 |

   a. Use the expected value approach to decide whether to market the new product.
   b. Because of the high dollar values involved, especially the possibility of a $100,000 loss, the marketing vice president has expressed some concern about the use of the expected value approach. As a consequence, if a utility analysis is performed, what is the appropriate lottery?
   c. Assume that the following indifference probabilities are assigned. Do the utilities reflect the behavior of a risk taker or a risk avoider?

| Profit | Indifference Probability($p$) |
|---|---|
| \$100,000 | 0.95 |
| \$ 50,000 | 0.70 |
| \$0 | 0.50 |
| −\$ 50,000 | 0.25 |

d. Use expected utility to make a recommended decision.
e. Should the decision maker feel comfortable with the final decision recommended by the analysis?

10. A television network has been receiving low ratings for its programs. Currently, management is considering two alternatives for the Monday night 8:00 P.M.–9:00 P.M. time slot: a western with a well-known star or a reality show. The percentages of viewing audience estimates depend on the degree of program acceptance. The relevant data are as follows:

| | Percentage of Viewing Audience | |
|---|---|---|
| **Program Acceptance** | **Comedy** | **Reality Show** |
| High | 30% | 40% |
| Moderate | 25% | 20% |
| Poor | 20% | 15% |

The probabilities associated with program acceptance levels are as follows:

| | Probability | |
|---|---|---|
| **Program Acceptance** | **Comedy** | **Reality Show** |
| High | 0.30 | 0.30 |
| Moderate | 0.60 | 0.40 |
| Poor | 0.10 | 0.30 |

a. Using the expected value approach, which program should the network choose?
b. For a utility analysis, what is the appropriate lottery?
c. Based on the lottery in part (b), assume that the network's program manager has assigned the following indifference probabilities. Based on the use of utility measures, which program would you recommend? Is the manager a risk taker or a risk avoider?

| Percentage of Audience | Indifference Probability ($p$) |
|---|---|
| 30% | 0.40 |
| 25% | 0.30 |
| 20% | 0.10 |

SELF test

11. Consider the following two-person, zero-sum game. Identify the pure strategy. What is the value of the game?

| | | Player B | | |
|---|---|---|---|---|
| | | $b_1$ | $b_2$ | $b_3$ |
| Player A | $a_1$ | 8 | 5 | 7 |
| | $a_2$ | 2 | 4 | 10 |

12. Two opposing armies, Red and Blue, must each decide whether to attack or defend. These decisions are made without knowledge of the opposing army's decision. The payoff table, in terms of value of property gained or lost for the Red Army, appears below. Any gains for the Red Army are losses for the Blue Army.

| | | Blue Army | |
|---|---|---|---|
| | | Attack | Defend |
| Red Army | Attack | 30 | 50 |
| | Defend | 40 | 0 |

   a. What is the optimal mixed strategy for the Red Army?
   b. What is the optimal mixed strategy for the Blue Army?

13. Two Indiana state senate candidates must decide which city to visit the day before the November election. The same four cities—Indianapolis, Evansville, Fort Wayne, and South Bend—are available for both candidates. These cities are listed as strategies 1 to 4 for each candidate. Travel plans must be made in advance, so the candidates must decide which city to visit prior to knowing the other candidate's plans. Values in the following table show thousands of voters for the Republican candidate based on the strategies selected by the two candidates. Which city should each candidate visit, and what is the value of the game?

| | | Democrat Candidate | | | |
|---|---|---|---|---|---|
| | | Indianapolis $b_1$ | Evansville $b_2$ | Fort Wayne $b_3$ | South Bend $b_4$ |
| Republican Candidate | Indianapolis, $a_1$ | 0 | −15 | −8 | 20 |
| | Evansville, $a_2$ | 30 | −5 | 5 | −10 |
| | Fort Wayne, $a_3$ | 10 | −25 | 0 | 20 |
| | South Bend, $a_4$ | 20 | 20 | 10 | 15 |

**SELF test**

14. In Section 5.5, we showed the following two-person, zero-sum game had a mixed strategy:

| | | Player B | | |
|---|---|---|---|---|
| | | $b_1$ | $b_2$ | $b_3$ |
| Player A | $a_1$ | 0 | −1 | 2 |
| | $a_2$ | 5 | 4 | −3 |
| | $a_3$ | 2 | 3 | −4 |

   a. Use dominance to reduce the game to a $2 \times 2$ game. Which strategies are dominated?
   b. Determine the optimal mixed strategy solution.
   c. What is the value of the game?

15. In a gambling game, Player A and Player B both have a \$1 and a \$5 bill. Each player selects one of the bills without the other player knowing the bill selected. Simultaneously they both reveal the bills selected. If the bills do not match, Player A wins Player B's bill. If the bills match, Player B wins Player A's bill.
    a. Develop the game theory table for this game. The values should be expressed as the gains (or losses) for Player A.
    b. Is there a pure strategy? Why or why not?
    c. Determine the optimal strategies and the value of this game. Does the game favor one player over the other?
    d. Suppose Player B decides to deviate from the optimal strategy and begins playing each bill 50% of the time. What should Player A do to improve Player A's winnings? Comment on why it is important to follow an optimal game theory strategy.

16. Two companies compete for a share of the soft drink market. Each worked with an advertising agency in order to develop alternative advertising strategies for the coming year. A variety of television advertisements, product promotions, in-store displays, and so on provides four different strategies for each company. The following table summarizes the projected change in market share for Company A once the two companies select their advertising strategy for the coming year. What is the optimal solution to this game for each of the players? What is the value of the game?

| | | Company B | | | |
|---|---|---|---|---|---|
| | | $b_1$ | $b_2$ | $b_3$ | $b_4$ |
| Company A | $a_1$ | 3 | 0 | 2 | 4 |
| | $a_2$ | 2 | −2 | 1 | 0 |
| | $a_3$ | 4 | 2 | 5 | 6 |
| | $a_4$ | −2 | 6 | −1 | 0 |

17. If a soccer game ends in a tie, it goes into a penalty-kick shootout in which each team chooses five players to take penalty kicks. The team that makes the most subsequent penalty kicks wins the game. In a penalty-kick shootout, the shooter and the keeper each decide simultaneously on a direction to move. They can choose left, right, or middle. These strategies yield the following game theory table, where the first value is the shooter's probability of scoring and the second value is the keeper's probability of stopping the shot:

| | | Keeper | | |
|---|---|---|---|---|
| | | **Left** | **Center** | **Right** |
| Shooter | **Left** | 0.35, 0.65 | 0.90, 0.10 | 0.85. 0.15 |
| | **Center** | 0.30, 0.70 | 0.25, 0.75 | 0.45, 0.55 |
| | **Right** | 0.95, 0.05 | 0.90, 0.10 | 0.30, 0.70 |

This is an example of a constant-sum game since each pair of entries in the game theory table sums to 1. This can be analyzed in the same manner as a zero-sum game.
    a. Use dominance to reduce the game to a 2 × 2 game. Which strategies are dominated?
    b. What is the solution to this game for the shooter and for the keeper?
    c. What is the shooter's expected probability of scoring?

## Case Problem—Utility, Game Theory, and Product Line Extension Decisions

ABC Inc. and XYZ Co. are the two dominant companies providing chargers, adapters, and other accessories for cell phones. Each of these Silicon Valley companies is developing a new line of smartphone accessories, and each has a choice of technologies to use for these accessories. Each company can choose to focus on older (cheaper) technology, recent (more expensive) technology, or cutting-edge (very expensive) technology. The share each company will gain or lose in the cell-phone accessories market depends on its technology choice and the technology choice of its competitor. The choice in technology investment must be made by each company before its competitor's choice is revealed.

ABC Inc. and XYZ Co. are both led by young, aggressive CEOs; ABC's CEO is Jack Webster and XYZ is run by Curtis Madsen. The CEOs of ABC and XYZ are each trying to determine the best technology in which to invest. The following tables provide market share values and indifference probability values, *p*, for both Jack Webster (ABC) and Curtis Madsen (XYZ). Jack and Curtis have identical indifference probability values.

| Market Share Gain (Loss) | Indifference Probability, *p* |
|---|---|
| 25 | — |
| 20 | 0.85 |
| 15 | 0.70 |
| 10 | 0.58 |
| 5 | 0.45 |
| 0 | 0.35 |
| –5 | 0.27 |
| –10 | 0.20 |
| –15 | 0.12 |
| –20 | 0.05 |
| –25 | — |

The following table shows the three possible technology investments (old, recent, and cutting edge) for ABC and XYZ and the resulting market share gain (or loss) for ABC, Inc. Because ABC and XYZ are the dominant companies in this market, whatever market share is gained by ABC is lost by XYZ and vice versa.

| | | XYZ, Co. | | |
|---|---|---|---|---|
| | | **Old** | **Recent** | **Cutting Edge** |
| | **Old** | 5 | −10 | −15 |
| **ABC, Inc.** | **Recent** | 10 | −5 | −10 |
| | **Cutting Edge** | 20 | 5 | −20 |

## Managerial Report

Perform an analysis of the best decision of technology investment for ABC Inc. Prepare a report you would provide to Jack Webster that summarizes your analysis and findings. Include the following:

1. A graph of the utility function for ABC, Inc. Explain whether you would characterize Jack Webster as a risk taker or a risk avoider and why.
2. Payoff tables for ABC, Inc. and XYZ, Co. using expected utilities for ABC and XYZ.
3. Recommendation for the best decision for ABC, Inc. and XYZ, Co. Is this a zero-sum game?
4. Detailed calculations and analysis to support your recommendation.
5. A discussion of how the expected utilities would change if ABC is taken over by a more conservative, risk-avoiding firm.