CHAPTER 7

Introduction to Linear Programming

CONTENTS

Linear programming is a problem-solving approach developed to help managers make decisions. Numerous applications of linear programming can be found in today's competitive business environment. For instance, IBM uses linear programming to perform capacity planning and to make capacity investment decisions for its semiconductor manufacturing operations. GE Capital uses linear programming to help determine optimal lease structuring. Marathon Oil Company uses linear programming for gasoline blending and to evaluate the economics of a new terminal or pipeline. The Q.M. in Action, Timber Harvesting Model at MeadWestvaco Corporation, provides another example of the use of linear programming. Later in the chapter another Q.M. in Action illustrates how the United States Navy uses linear programming to reduce fuel consumption for its ships.

To illustrate some of the properties that all linear programming problems have in common, consider the following typical applications:

1. A manufacturer wants to develop a production schedule and an inventory policy that will satisfy sales demand in future periods. Ideally, the schedule and policy will enable the company to satisfy demand and at the same time *minimize* the total production and inventory costs.
2. A financial analyst must select an investment portfolio from a variety of stock and bond investment alternatives. The analyst would like to establish the portfolio that *maximizes* the return on investment.
3. A marketing manager wants to determine how best to allocate a fixed advertising budget among alternative advertising media such as radio, television, newspaper, and magazine. The manager would like to determine the media mix that *maximizes* advertising effectiveness.
4. A company has warehouses in a number of locations. Given specific customer demands, the company would like to determine how much each warehouse should ship to each customer so that total transportation costs are *minimized.*

These examples are only a few of the situations in which linear programming has been used successfully, but they illustrate the diversity of linear programming applications. A close scrutiny reveals one basic property they all have in common. In each example, we were concerned with *maximizing* or *minimizing* some quantity. In example 1, the manufacturer wanted to minimize costs; in example 2, the financial analyst wanted to maximize return on investment; in example 3, the marketing manager wanted to maximize advertising effectiveness; and in example 4, the company wanted to minimize total transportation costs. In all linear programming problems, the maximization or minimization of some quantity is the objective.

Q.M. *in* ACTION

*TIMBER HARVESTING MODEL AT MEADWESTVACO CORPORATION**

MeadWestvaco Corporation is a major producer of premium papers for periodicals, books, commercial printing, and business forms. The company also produces pulp and lumber, designs and manufactures packaging systems for beverage and other consumables markets, and is a world leader in the production of coated board and shipping containers. Quantitative analyses at MeadWestvaco are developed and implemented by the company's Decision Analysis Department. The department assists decision makers by providing them with analytical tools of quantitative methods as well as personal analysis and recommendations.

(continued)

*Based on information provided by Dr. Edward P. Winkofsky of MeadWestvaco Corporation.

MeadWestvaco uses quantitative models to assist with the long-range management of the company's timberland. Through the use of large-scale linear programs, timber harvesting plans are developed to cover a substantial time horizon. These models consider wood market conditions, mill pulpwood requirements, harvesting capacities, and general forest management principles. Within these constraints, the model arrives at an optimal harvesting and purchasing schedule based on discounted cash flow. Alternative schedules reflect changes in the various assumptions concerning forest growth, wood availability, and general economic conditions.

Quantitative methods are also used in the development of the inputs for the linear programming models. Timber prices and supplies as well as mill requirements must be forecast over the time horizon, and advanced sampling techniques are used to evaluate land holdings and to project forest growth. The harvest schedule is then developed using quantitative methods.

Linear programming was initially referred to as "programming in a linear structure." In 1948 Tjalling Koopmans suggested to George Dantzig that the name was much too long: Koopman's suggestion was to shorten it to linear programming. George Dantzig agreed and the field we now know as linear programming was named.

All linear programming problems also have a second property: restrictions or **constraints** that limit the degree to which the objective can be pursued. In the first example, the manufacturer is restricted by constraints requiring product demand to be satisfied and by the constraints limiting production capacity. The financial analyst's portfolio problem is constrained by the total amount of investment funds available and the maximum amounts that can be invested in each stock or bond. The marketing manager's media selection decision is constrained by a fixed advertising budget and the availability of the various media. In the transportation problem, the minimum-cost shipping schedule is constrained by the supply of product available at each warehouse. Thus, constraints are another general feature of every linear programming problem.

7.1 A Simple Maximization Problem

RMC, Inc., is a small firm that produces a variety of chemical-based products. In a particular production process, three raw materials are used to produce two products: a fuel additive and a solvent base. The fuel additive is sold to oil companies and is used in the production of gasoline and related fuels. The solvent base is sold to a variety of chemical firms and is used in both home and industrial cleaning products. The three raw materials are blended to form the fuel additive and solvent base as indicated in Table 7.1, which shows that a ton of fuel additive is a mixture of 0.4 tons of material 1 and 0.6 tons of material 3. A ton of solvent base is a mixture of 0.5 tons of material 1, 0.2 tons of material 2, and 0.3 tons of material 3.

TABLE 7.1 MATERIAL REQUIREMENTS PER TON FOR THE RMC PROBLEM

	Product	
	Fuel Additive	**Solvent Base**
Material 1	0.4	0.5
Material 2		0.2
Material 3	0.6	0.3

0.6 tons of material 3 is used in each ton of fuel additive

RMC's production is constrained by a limited availability of the three raw materials. For the current production period, RMC has available the following quantities of each raw material:

Material	Amount Available for Production
Material 1	20 tons
Material 2	5 tons
Material 3	21 tons

It is important to understand that we are maximizing profit contribution, not profit. Overhead and other shared costs must be deducted before arriving at a profit figure.

Because of spoilage and the nature of the production process, any materials not used for current production are useless and must be discarded.

The accounting department analyzed the production figures, assigned all relevant costs, and arrived at prices for both products that will result in a profit contribution[1] of $40 for every ton of fuel additive produced and $30 for every ton of solvent base produced. Let us now use linear programming to determine the number of tons of fuel additive and the number of tons of solvent base to produce in order to maximize total profit contribution.

Problem Formulation

Problem formulation is the process of translating a verbal statement of a problem into a mathematical statement. The mathematical statement of the problem is referred to as a **mathematical model**. Developing an appropriate mathematical model is an art that can only be mastered with practice and experience. Even though every problem has at least some unique features, most problems also have many common or similar features. As a result, some general guidelines for developing a mathematical model can be helpful. We will illustrate these guidelines by developing a mathematical model for the RMC problem.

Understand the Problem Thoroughly The RMC problem is relatively easy to understand. RMC wants to determine how much of each product to produce in order to maximize the total contribution to profit. The number of tons available for the three materials that are required to produce the two products will limit the number of tons of each product that can be produced. More complex problems will require more work in order to understand the problem. However, understanding the problem thoroughly is the first step in developing any mathematical model.

Describe the Objective RMC's objective is to maximize the total contribution to profit.

Describe Each Constraint Three constraints limit the number of tons of fuel additive and the number of tons of solvent base that can be produced.

Constraint 1: The number of tons of material 1 used must be less than or equal to the 20 tons available.

Constraint 2: The number of tons of material 2 used must be less than or equal to the 5 tons available.

Constraint 3: The number of tons of material 3 used must be less than or equal to the 21 tons available.

[1]From an accounting perspective, profit contribution is more correctly described as the contribution margin per ton; overhead and other shared costs have not been allocated to the fuel additive and solvent base costs.

Define the Decision Variables The **decision variables** are the controllable inputs in the problem. For the RMC problem the two decision variables are (1) the number of tons of fuel additive produced, and (2) the number of tons of solvent base produced. In developing the mathematical model for the RMC problem, we will use the following notation for the decision variables:

$$F = \text{number of tons of fuel additive}$$
$$S = \text{number of tons of solvent base}$$

Write the Objective in Terms of the Decision Variables RMC's profit contribution comes from the production of F tons of fuel additive and S tons of solvent base. Because RMC makes \$40 for every ton of fuel additive produced and \$30 for every ton of solvent base produced, the company will make \$40F from the production of the fuel additive and \$30S from the production of the solvent base. Thus,

$$\text{Total profit contribution} = 40F + 30S$$

Because the objective—maximize total profit contribution—is a function of the decision variables F and S, we refer to $40F + 30S$ as the **objective function**. Using "Max" as an abbreviation for maximize, we can write RMC's objective as follows:

$$\text{Max } 40F + 30S \qquad \textbf{(7.1)}$$

Write the Constraints in Terms of the Decision Variables

Constraint 1:

$$\text{Tons of material 1 used} \leq \text{Tons of material 1 available}$$

Every ton of fuel additive that RMC produces will use 0.4 tons of material 1. Thus, 0.4F tons of material 1 is used to produce F tons of fuel additive. Similarly, every ton of solvent base that RMC produces will use 0.5 tons of material 1. Thus, 0.5S tons of material 1 is used to produce S tons of solvent base. Therefore, the number of tons of material 1 used to produce F tons of fuel additive and S tons of solvent base is

$$\text{Tons of material 1 used} = 0.4F + 0.5S$$

Because 20 tons of material 1 are available for use in production, the mathematical statement of constraint 1 is

$$0.4F + 0.5S \leq 20 \qquad \textbf{(7.2)}$$

Constraint 2:

$$\text{Tons of material 2 used} \leq \text{Tons of material 2 available}$$

Fuel additive does not use material 2. However, every ton of solvent base that RMC produces will use 0.2 tons of material 2. Thus, 0.2S tons of material 2 is used to produce S tons of solvent base. Therefore, the number of tons of material 2 used to produce F tons of fuel additive and S tons of solvent base is

$$\text{Tons of material 2 used} = 0.2S$$

Because 5 tons of material 2 are available for production, the mathematical statement of constraint 2 is

$$0.2S \leq 5 \tag{7.3}$$

Constraint 3:

Tons of material 3 used ≤ Tons of material 3 available

Every ton of fuel additive RMC produces will use 0.6 tons of material 3. Thus, $0.6F$ tons of material 1 is used to produce F tons of fuel additive. Similarly, every ton of solvent base RMC produces will use 0.3 tons of material 3. Thus, $0.3S$ tons of material 1 is used to produce S tons of solvent base. Therefore, the number of tons of material 3 used to produce F tons of fuel additive and S tons of solvent base is

$$\text{Tons of material 3 used} = 0.6F + 0.3S$$

Because 21 tons of material 3 are available for production, the mathematical statement of constraint 3 is

$$0.6F + 0.3S \leq 21 \tag{7.4}$$

Add the Nonnegativity Constraints RMC cannot produce a negative number of tons of fuel additive or a negative number of tons of solvent base. Therefore, **nonnegativity constraints** must be added to prevent the decision variables F and S from having negative values. These nonnegativity constraints are

$$F \geq 0 \text{ and } S \geq 0$$

Nonnegativity constraints are a general feature of many linear programming problems and may be written in the abbreviated form:

$$F, S \geq 0 \tag{7.5}$$

Mathematical Model for the RMC Problem

Problem formulation is now complete. We have succeeded in translating the verbal statement of the RMC problem into the following mathematical model:

$$\begin{aligned}
&\text{Max } 40F + 30S \\
&\text{Subject to (s.t.)} \\
&\qquad 0.4F + 0.5S \leq 20 \quad \text{Material 1} \\
&\qquad\qquad\quad 0.2S \leq 5 \quad \text{Material 2} \\
&\qquad 0.6F + 0.3S \leq 21 \quad \text{Material 3} \\
&\qquad F, S \geq 0
\end{aligned}$$

Our job now is to find the product mix (i.e., the combination of F and S) that satisfies all the constraints and, at the same time, yields a maximum value for the objective function. Once these values of F and S are calculated, we will have found the optimal solution to the problem.

This mathematical model of the RMC problem is a **linear program**. The RMC problem has an objective and constraints that, as we said earlier, are common properties of all *linear* programs. But what is the special feature of this mathematical model that makes it a linear program? The special feature that makes it a linear program is that the objective function and all constraint functions (the left-hand sides of the constraint inequalities) are linear functions of the decision variables.

Mathematical functions in which each variable appears in a separate term and is raised to the first power are called **linear functions**. The objective function ($40F + 30S$) is linear because each decision variable appears in a separate term and has an exponent of 1. The amount of material 1 used ($0.4F + 0.5S$) is also a linear function of the decision variables for the same reason. Similarly, the functions on the left-hand side of the material 2 and material 3 constraint inequalities (the constraint functions) are also linear functions. Thus, the mathematical formulation is referred to as a linear program.

Try Problem 1 to test your ability to recognize the types of mathematical relationships that can be found in a linear program.

Linear *programming* has nothing to do with computer programming. The use of the word *programming* here means "choosing a course of action." Linear programming involves choosing a course of action when the mathematical model of the problem contains only linear functions.

NOTES AND COMMENTS

1. The three assumptions necessary for a linear programming model to be appropriate are proportionality, additivity, and divisibility. *Proportionality* means that the contribution to the objective function and the amount of resources used in each constraint are proportional to the value of each decision variable. *Additivity* means that the value of the objective function and the total resources used can be found by summing the objective function contribution and the resources used for all decision variables. *Divisibility* means that the decision variables are continuous. The divisibility assumption plus the nonnegativity constraints mean that decision variables can take on any value greater than or equal to zero.
2. Quantitative analysts formulate and solve a variety of mathematical models that contain an objective function and a set of constraints. Models of this type are referred to as *mathematical programming models*. Linear programming models are a special type of mathematical programming model in that the objective function and all constraint functions are linear.

7.2 Graphical Solution Procedure

A linear programming problem involving only two decision variables can be solved using a graphical solution procedure. Let us begin the graphical solution procedure by developing a graph that displays the possible solutions (F and S values) for the RMC problem. The graph in Figure 7.1 has values of F on the horizontal axis and values of S on the vertical axis. Any point on the graph can be identified by its F and S values, which indicate the position of the point along the horizontal and vertical axes, respectively. Thus, every point on the graph corresponds to a possible solution. The solution of $F = 0$ and $S = 0$ is referred to as the origin. Because both F and S must be nonnegative, the graph in Figure 7.1 only displays solutions where $F \geq 0$ and $S \geq 0$.

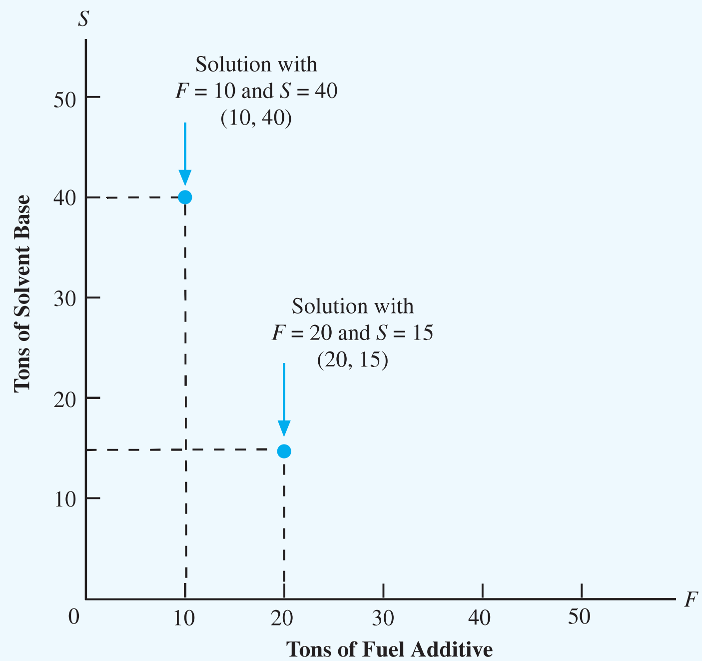
Earlier we determined that the inequality representing the material 1 constraint was

$$0.4F + 0.5S \leq 20$$

To show all solutions that satisfy this relationship, we start by graphing the line corresponding to the equation

$$0.4F + 0.5S = 20$$

FIGURE 7.1 GRAPH SHOWING TWO SOLUTIONS FOR THE TWO-VARIABLE RMC PROBLEM



We graph this equation by identifying two points that satisfy this equation and then drawing a line through the points. Setting $F = 0$ and solving for S gives $0.5S = 20$, or $S = 40$; hence the solution ($F = 0$, $S = 40$) satisfies the preceding equation. To find a second solution satisfying this equation, we set $S = 0$ and solve for F. Doing so, we obtain $0.4F = 20$, or $F = 50$. Thus, a second solution satisfying the equation is ($F = 50$, $S = 0$). With these two points, we can now graph the line. This line, called the *material 1 constraint line,* is shown in Figure 7.2.

Recall that the inequality representing the material 1 constraint is

$$0.4F + 0.5S \leq 20$$

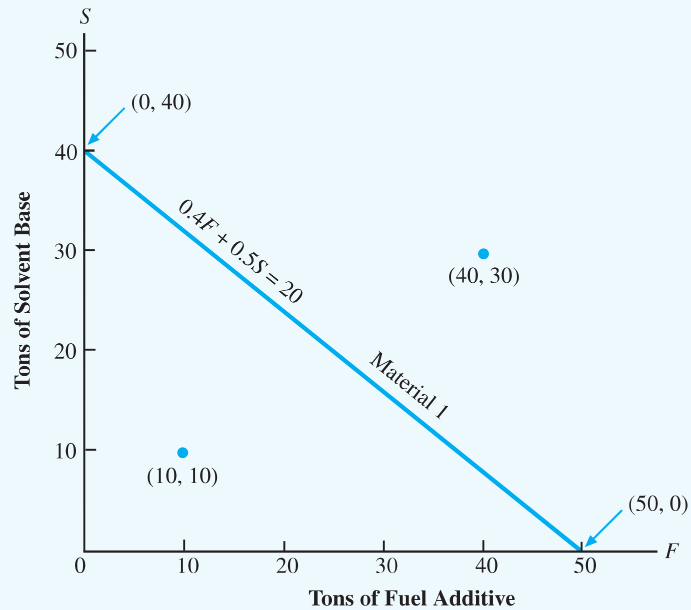
Can you identify all the solutions that satisfy this constraint? First, note that any point on the line $0.4F + 0.5S = 20$ must satisfy the constraint. But where are the solutions satisfying $0.4F + 0.5S < 20$? Consider two solutions ($F = 10$, $S = 10$) and ($F = 40$, $S = 30$). Figure 7.2 shows that the first solution is on the same side of the constraint line as the origin while the second solution is on the side of the constraint line opposite of the origin. Which of these solutions satisfies the material 1 constraint? For ($F = 10$, $S = 10$) we have

$$0.4F + 0.5S = 0.4(10) + 0.5(10) = 9$$

Because 9 tons is less than the 20 tons of material 1 available, the $F = 10$, $S = 10$ solution satisfies the constraint. For $F = 40$ and $S = 30$ we have

$$0.4F + 0.5S = 0.4(40) + 0.5(30) = 31$$

FIGURE 7.2 MATERIAL 1 CONSTRAINT LINE



The 31 tons is greater than the 20 tons available, so the $F = 40$, $S = 30$ solution does not satisfy the constraint.

You should now be able to graph a constraint line and find the solution points that satisfy the constraint. Try Problem 2.

If a particular solution satisfies the constraint, all other solutions on the same side of the constraint line will also satisfy the constraint. If a particular solution does not satisfy the constraint, all other solutions on the same side of the constraint line will not satisfy the constraint. Thus, you need to evaluate only one solution to determine which side of a constraint line provides solutions that will satisfy the constraint. The shaded area in Figure 7.3 shows all the solutions that satisfy the material 1 constraint.

Next let us identify all solutions that satisfy the material 2 constraint:

$$0.2S \leq 5$$

We start by drawing the constraint line corresponding to the equation $0.2S = 5$. Because this equation is equivalent to the equation $S = 25$, we simply draw a line whose S value is 25 for every value of F; this line is parallel to and 25 units above the horizontal axis. Figure 7.4 shows the line corresponding to the material 2 constraint. Following the approach we used for the material 1 constraint, we realize that only solutions on or below the line will satisfy the material 2 constraint. Thus, in Figure 7.4 the shaded area corresponds to the solutions that satisfy the material 2 constraint.

Similarly, we can determine the solutions that satisfy the material 3 constraint. Figure 7.5 shows the result. For practice, try to graph the feasible solutions that satisfy the material 3 constraint and determine whether your result agrees with that shown in Figure 7.5.

We now have three separate graphs showing the solutions that satisfy each of the three constraints. In a linear programming problem, we need to identify the solutions that satisfy

FIGURE 7.3 SOLUTIONS THAT SATISFY THE MATERIAL 1 CONSTRAINT

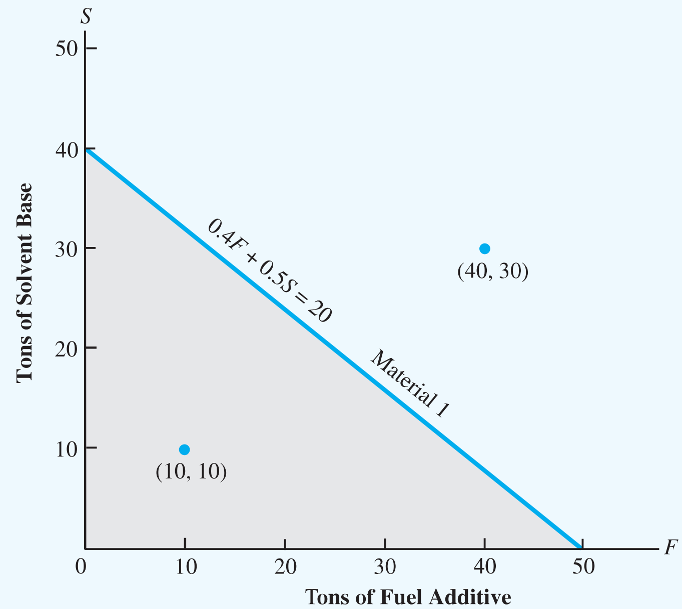


FIGURE 7.4 SOLUTIONS THAT SATISFY THE MATERIAL 2 CONSTRAINT

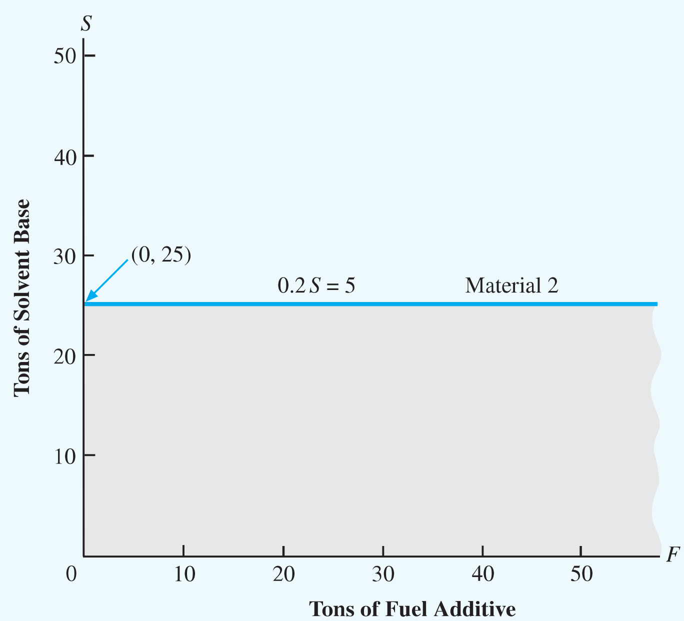
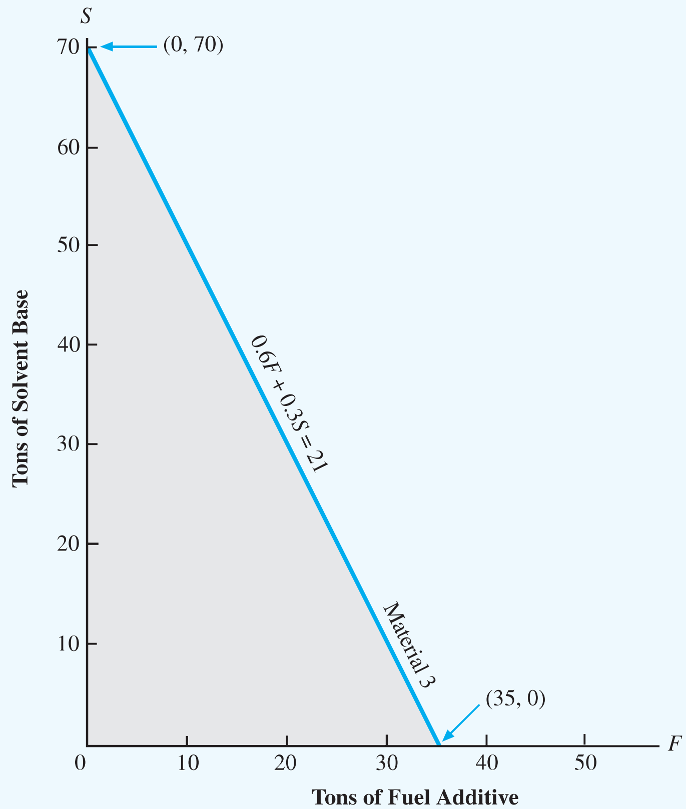


FIGURE 7.5 SOLUTIONS THAT SATISFY THE MATERIAL 3 CONSTRAINT



all the constraints *simultaneously.* To find these solutions, we can draw the three constraints on one graph and observe the region containing the points that do in fact satisfy all the constraints simultaneously.

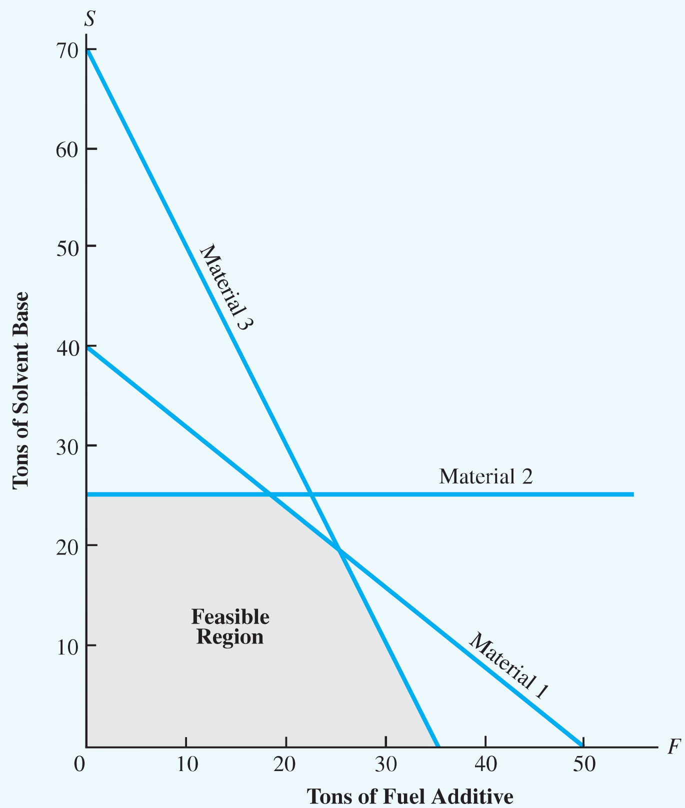
The graphs in Figures 7.3, 7.4, and 7.5 can be superimposed to obtain one graph with all three constraints. Figure 7.6 shows this combined constraint graph. The shaded region in this figure includes every solution point that satisfies all the constraints simultaneously. Because solutions that satisfy all the constraints simultaneously are termed **feasible solutions**, the shaded region is called the *feasible solution region,* or simply the **feasible region**. Any point on the boundary of the feasible region, or within the feasible region, is a *feasible solution point* for the linear programming problem.

Can you now find the feasible region given several constraints? Try Problem 7.

Now that we have identified the feasible region, we are ready to proceed with the graphical solution method and find the optimal solution to the RMC problem. Recall that the optimal solution for a linear programming problem is the feasible solution that provides the best possible value of the objective function. Let us start the optimizing step of the graphical solution procedure by redrawing the feasible region on a separate graph. Figure 7.7 shows the graph.

One approach to finding the optimal solution would be to evaluate the objective function for each feasible solution; the optimal solution would then be the one yielding the

FIGURE 7.6 FEASIBLE REGION FOR THE RMC PROBLEM



largest value. The difficulty with this approach is that the infinite number of feasible solutions makes evaluating all feasible solutions impossible. Hence, this trial-and-error procedure cannot be used to identify the optimal solution.

Rather than trying to compute the profit contribution for each feasible solution, we select an arbitrary value for profit contribution and identify all the feasible solutions that yield the selected value. For example, what feasible solutions provide a profit contribution of \$240? These solutions are given by the values of F and S in the feasible region that will make the objective function

$$40F + 30S = 240$$

This expression is simply the equation of a line. Thus all feasible solutions (F, S) yielding a profit contribution of \$240 must be on the line. We learned earlier in this section how to graph a constraint line. The procedure for graphing the profit or objective function line is the same. Letting $F = 0$, we see that S must be 8; thus the solution point ($F = 0$, $S = 8$) is on the line. Similarly, by letting $S = 0$ we see that the solution point ($F = 6$, $S = 0$) is also on the line. Drawing the line through these two points identifies all the solutions that have a profit contribution of \$240. A graph of this profit line is presented in Figure 7.8. The graph shows that an infinite number of feasible production combinations will provide a \$240 profit contribution.

FIGURE 7.7 FEASIBLE REGION FOR THE RMC PROBLEM

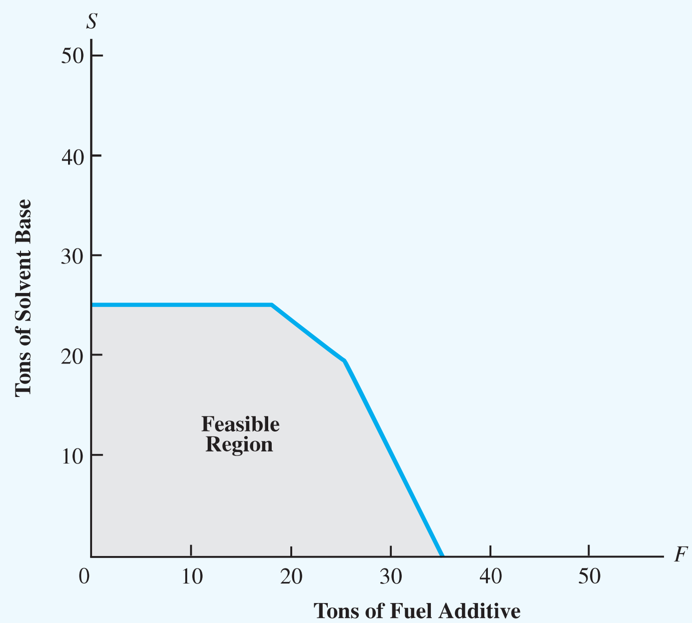
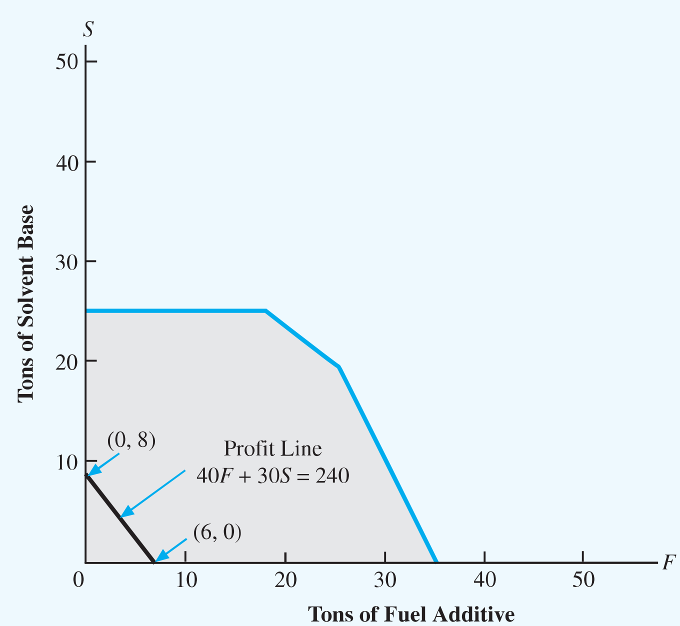


FIGURE 7.8 $240 PROFIT LINE FOR THE RMC PROBLEM



The objective is to find the feasible solution yielding the highest profit contribution, so we proceed by selecting higher profit contributions and finding the solutions that yield the stated values. For example, what solutions provide a profit contribution of $720? What solutions provide a profit contribution of $1200? To answer these questions, we must find the F and S values that are on the profit lines:

$$40F + 30S = 720 \text{ and } 40F + 30S = 1200$$

Using the previous procedure for graphing profit and constraint lines, we graphed the $720 and $1200 profit lines presented in Figure 7.9. Not all solution points on the $1200 profit line are in the feasible region, but at least some points on the line are; thus, we can obtain a feasible solution that provides a $1200 profit contribution.

Can we find a feasible solution yielding an even higher profit contribution? Look at Figure 7.9 and make some general observations about the profit lines. You should be able to identify the following properties: (1) The profit lines are *parallel* to each other, and (2) profit lines with higher profit contributions are farther from the origin.

Because the profit lines are parallel and higher profit lines are farther from the origin, we can obtain solutions that yield increasingly higher values for the objective function by continuing to move the profit line farther from the origin but keeping it parallel to the other profit lines. However, at some point any further outward movement will place the profit line entirely outside the feasible region. Because points outside the feasible region are unacceptable, the point in the feasible region that lies on the highest profit line is an optimal solution to the linear program.

You should now be able to identify the optimal solution point for the RMC problem. Use a ruler and move the profit line as far from the origin as you can. What is the last point in the feasible region? This point, which is the optimal solution, is shown graphically in Figure 7.10. The optimal values for the decision variables are the F and S values at this point.

FIGURE 7.9 SELECTED PROFIT LINES FOR THE RMC PROBLEM

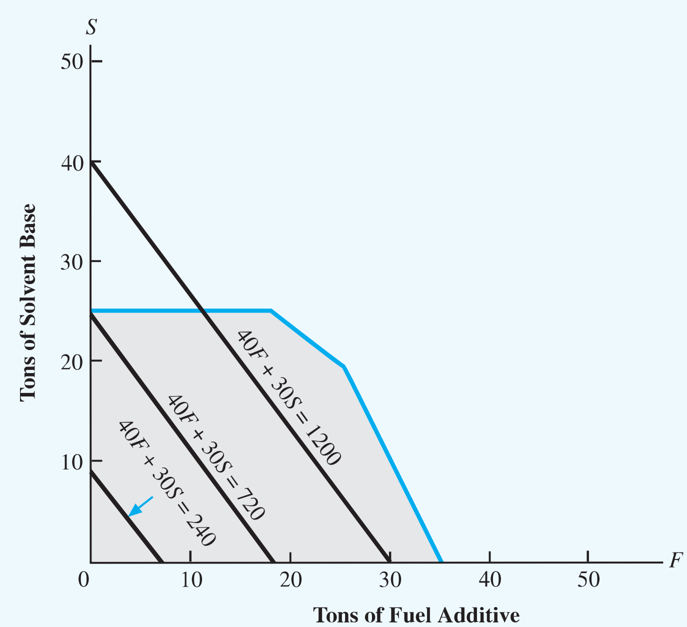
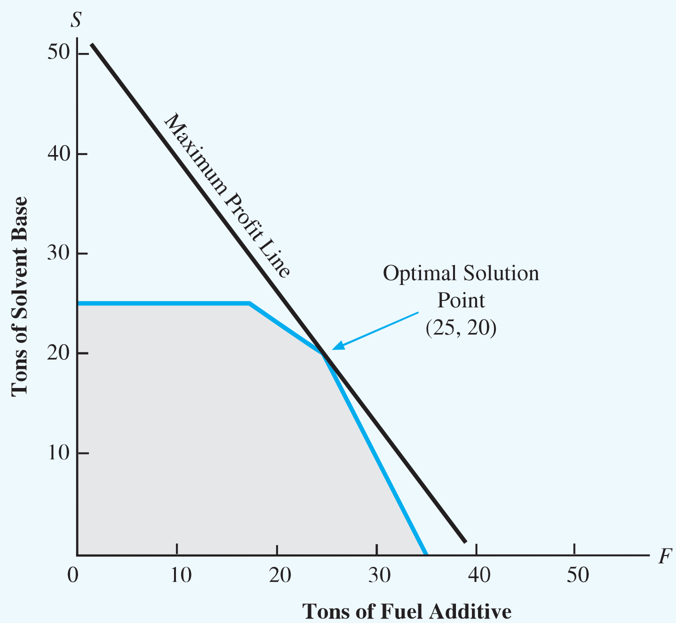


FIGURE 7.10 OPTIMAL SOLUTION FOR THE RMC PROBLEM



Depending on the accuracy of your graph, you may or may not be able to determine the exact optimal values of F and S directly from the graph. However, refer to Figure 7.6 and note that the optimal solution point for the RMC example is at the *intersection* of the material 1 and material 3 constraint lines. That is, the optimal solution is on both the material 1 constraint line,

$$0.4F + 0.5S = 20 \tag{7.6}$$

and the material 3 constraint line,

$$0.6F + 0.3S = 21 \tag{7.7}$$

Thus, the values of the decision variables F and S must satisfy both equations (7.6) and (7.7) simultaneously. Using (7.6) and solving for F gives

$$0.4F = 20 - 0.5S$$

or

$$F = 50 - 1.25S \tag{7.8}$$

Substituting this expression for F into equation (7.7) and solving for S yields

$$\begin{aligned} 0.6(50 - 1.25S) + 0.3S &= 21 \\ 30 - 0.75S + 0.3S &= 21 \\ -0.45S &= -9 \\ S &= 20 \end{aligned}$$

Substituting $S = 20$ in equation (7.8) and solving for F provides

$$F = 50 - 1.25(20)$$
$$= 50 - 25 = 25$$

Although the optimal solution to the RMC problem consists of integer values for the decision variables, this result will not always be the case.

Thus, the exact location of the optimal solution point is $F = 25$ and $S = 20$. This solution point provides the optimal production quantities for RMC at 25 tons of fuel additive and 20 tons of solvent base and yields a profit contribution of $40(25) + 30(20) = \$1600$.

For a linear programming problem with two decision variables, you can determine the exact values of the decision variables at the optimal solution by first using the graphical procedure to identify the optimal solution point and then solving the two simultaneous equations associated with this point.

A Note on Graphing Lines

An important aspect of the graphical method is the ability to graph lines showing the constraints and the objective function of the linear program. The procedure we used for graphing the equation of a line is to find any two points satisfying the equation and then draw the line through the two points. For the RMC constraints, the two points were easily found by setting $F = 0$ and solving the constraint equation for S. Then we set $S = 0$ and solved for F. For the material 1 constraint line

$$0.4F + 0.5S = 20$$

this procedure identified the two points ($F = 0$, $S = 40$) and ($F = 50$, $S = 0$). The material 1 constraint line was then graphed by drawing a line through these two points.

Try Problem 10 to test your ability to use the graphical solution procedure to identify the optimal solution and find the exact values of the decision variables at the optimal solution.

All constraints and objective function lines in two-variable linear programs can be graphed if two points on the line can be identified. However, finding the two points on the line is not always as easy as shown in the RMC problem. For example, suppose a company manufactures two models of a small hand-held computer: the Professional (P) and the Assistant (A). Management needs 50 units of the Professional model for its own sales force and expects sales of the remaining Professionals to be less than or equal to 50% of the sales of the Assistant. A constraint enforcing this requirement is

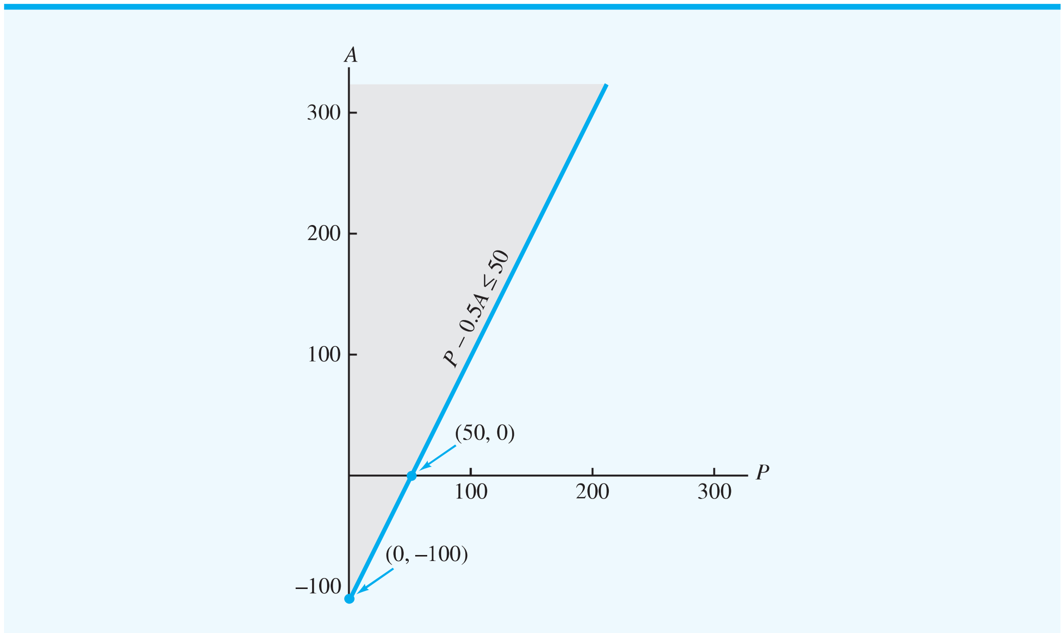
$$P - 50 \leq 0.5A$$

or

$$P - 0.5A \leq 50$$

Using the equality form of the constraint and setting $P = 0$, we find that the point ($P = 0$, $A = -100$) is on the constraint line. Setting $A = 0$, we find a second point ($P = 50$, $A = 0$) on the constraint line. If we have drawn only the nonnegative ($P \geq 0$, $A \geq 0$) portion of the graph, the first point ($P = 0$, $A = -100$) cannot be plotted because $A = -100$ is not on the graph. Whenever we have two points on the line, but one or both of the points cannot be plotted in the nonnegative portion of the graph, the simplest approach is to enlarge the graph. In this example, the point ($P = 0$, $A = -100$) can be plotted by extending the graph to include the negative A axis. Once both points satisfying the constraint equation have been located, the line can be drawn. The constraint line and the solutions that satisfy the constraint $P - 0.5A \leq 50$ are shown in Figure 7.11.

FIGURE 7.11 SOLUTIONS THAT SATISFY THE CONSTRAINT $P - 0.5A \leq 50$



As another example, consider a problem involving two decision variables, R and T. Suppose that the number of units of R produced has to be at least equal to the number of units of T produced. A constraint enforcing this requirement is

$$R \geq T$$

or

$$R - T \geq 0$$

Can you graph a constraint line when the origin is on the constraint line? Try Problem 5.

To find all solutions satisfying the constraint as an equality, we first set $R = 0$ and solve for T. This result shows that the origin ($T = 0$, $R = 0$) is on the constraint line. Setting $T = 0$ and solving for R provides the same point. However, we can obtain a second point on the line by setting T equal to any value other than zero and then solving for R. For instance, setting $T = 100$ and solving for R, we find that the point ($T = 100$, $R = 100$) is on the line. With the two points ($R = 0$, $T = 0$) and ($R = 100$, $T = 100$), the constraint line $R - T = 0$ and the solutions that satisfy the constraint $R - T \geq 0$ can be plotted as shown in Figure 7.12.

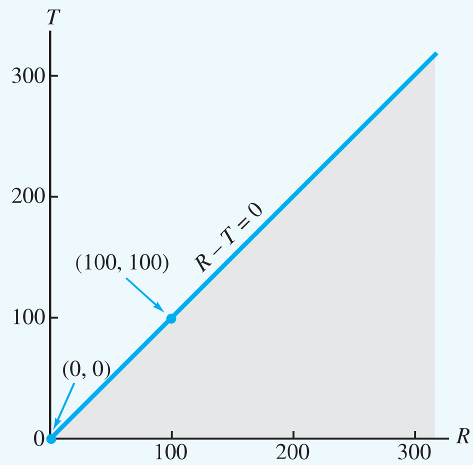
Summary of the Graphical Solution Procedure for Maximization Problems

For additional practice in using the graphical solution procedure, try Problem 24.

As we have seen, the graphical solution procedure is a method for solving two-variable linear programming problems such as the RMC problem. The steps of the graphical solution procedure for a maximization problem are summarized here:

1. Prepare a graph for each constraint that shows the solutions that satisfy the constraint.
2. Determine the feasible region by identifying the solutions that satisfy all the constraints simultaneously.

FIGURE 7.12 FEASIBLE SOLUTIONS FOR THE CONSTRAINT $R - T \geq 0$



3. Draw an objective function line showing the values of the decision variables that yield a specified value of the objective function.
4. Move parallel objective function lines toward larger objective function values until further movement would take the line completely outside the feasible region.
5. Any feasible solution on the objective function line with the largest value is an optimal solution.

Slack Variables

In addition to the optimal solution and its associated profit contribution, the RMC managers will want information about the production requirements for the three materials. We can determine this information by substituting the optimal solution values ($F = 25$, $S = 20$) into the constraints of the linear program.

Constraint	Tons Required for $F = 25$, $S = 20$ Tons	Tons Available	Unused Tons
Material 1	$0.4(25) + 0.5(20) = 20$	20	0
Material 2	$0.2(20) = 4$	5	1
Material 3	$0.6(25) + 0.3(20) = 21$	21	0

Thus, the optimal solution tells management that the production of 25 tons of fuel additive and 20 tons of solvent base will require all available material 1 and material 3 but only 4 of the 5 tons of material 2. The 1 ton of unused material 2 is referred to as *slack*. In linear programming terminology, any unused or idle capacity for a $\leq$ constraint is referred to as the *slack associated with the constraint*. Thus, the material 2 constraint has a slack of 1 ton.

Can you identify the slack associated with a constraint? Try Problem 24, part (e).

Often variables, called **slack variables**, are added to the formulation of a linear programming problem to represent the slack, or unused capacity, associated with a constraint. Unused capacity makes no contribution to profit, so slack variables have coefficients of zero in the objective function. More generally, slack variables represent the difference between

Can you write a linear program in standard form? Try Problem 18.

the right-hand side and the left-hand side of a ≤ constraint. After the addition of three slack variables, denoted S_1, S_2, and S_3, the mathematical model of the RMC problem becomes

$$
\begin{aligned}
\text{Max} \quad & 40F + 30S + 0S_1 + 0S_2 + 0S_3 \\
\text{s.t.} \quad & \\
& 0.4F + 0.5S + 1S_1 = 20 \\
& 0.2S + 1S_2 = 5 \\
& 0.6F + 0.3S + 1S_3 = 21 \\
& F, S, S_1, S_2, S_3 \geq 0
\end{aligned}
$$

Whenever a linear program is written in a form with all the constraints expressed as equalities, it is said to be written in **standard form**.

Referring to the standard form of the RMC problem, we see that at the optimal solution ($F = 25$, $S = 20$) the values for the slack variables are

Constraint	Value of Slack Variable
Material 1	$S_1 = 0$
Material 2	$S_2 = 1$
Material 3	$S_3 = 0$

Could we have used the graphical analysis to provide some of the previous information? The answer is yes. By finding the optimal solution in Figure 7.6, we see that the material 1 constraint and the material 3 constraint restrict, or *bind*, the feasible region at this point. Thus, the optimal solution requires the use of all of these two resources. In other words, the graph shows that at the optimal solution material 1 and material 3 will have zero slack. But, because the material 2 constraint is not binding the feasible region at the optimal solution, we can expect some slack for this resource.

Recognizing redundant constraints is easy with the graphical solution method. In problems with more than two decision variables, however, redundant constraints usually will not be apparent.

Finally, some linear programs may have one or more constraints that do not affect the feasible region; that is, the feasible region remains the same whether or not the constraint is included in the problem. Because such a constraint does not affect the feasible region and thus cannot affect the optimal solution, it is called a **redundant constraint**. Redundant constraints can be dropped from the problem without having any effect on the optimal solution. However, in most linear programming problems redundant constraints are not discarded because they are not immediately recognizable as being redundant. The RMC problem had no redundant constraints because each constraint had an effect on the feasible region.

NOTES AND COMMENTS

1. In the standard form representation of a linear program, the objective function coefficients for the slack variables are zero. This condition implies that slack variables, which represent unused resources, do not affect the value of the objective function. However, in some applications, some or all of the unused resources can be sold and contribute to profit. In such cases the corresponding slack variables become decision variables representing the amount of resources to be sold. For each of these variables, a nonzero coefficient in the objective function would reflect the profit associated with selling a unit of the corresponding resource.
2. Redundant constraints do not affect the feasible region; as a result they can be removed from a linear programming model without affecting the optimal solution. However, if the linear programming model is to be resolved later, changes in some of the data might change a previously redundant constraint into a binding constraint. Thus, we recommend keeping all constraints in the linear programming model even though one or more of the constraints may be redundant.

7.3 Extreme Points and the Optimal Solution

Suppose that the profit contribution for 1 ton of solvent base increases from \$30 to \$60 while the profit contribution for 1 ton of fuel additive and all the constraints remain unchanged. The complete linear programming model of this new problem is identical to the mathematical model in Section 7.2, except for the revised objective function:

$$\text{Max } 40F + 60S$$

How does this change in the objective function affect the optimal solution to the RMC problem? Figure 7.13 shows the graphical solution of the RMC problem with the revised objective function. Note that because the constraints do not change, the feasible region remains unchanged. However, the profit lines must be altered to reflect the new objective function.

By moving the profit line in a parallel manner away from the origin, we find the optimal solution as shown in Figure 7.13. The values of the decision variables at this point are $F = 18.75$ and $S = 25$. The increased profit for the solvent base caused a change in the optimal solution. In fact, as you might suspect, we cut back the production of the lower profit fuel additive and increase the production of the higher profit solvent base.

What do you notice about the location of the optimal solutions in the linear programming problems that we solved thus far? Look closely at the graphical solutions in Figures 7.10 and 7.13. An important observation that you should be able to make is that the optimal solutions occur at one of the vertices, or "corners," of the feasible region. In linear programming terminology these vertices are referred to as the **extreme points** of the feasible region. Thus,

FIGURE 7.13 OPTIMAL SOLUTION FOR THE RMC PROBLEM WITH AN OBJECTIVE FUNCTION OF $40F + 60S$

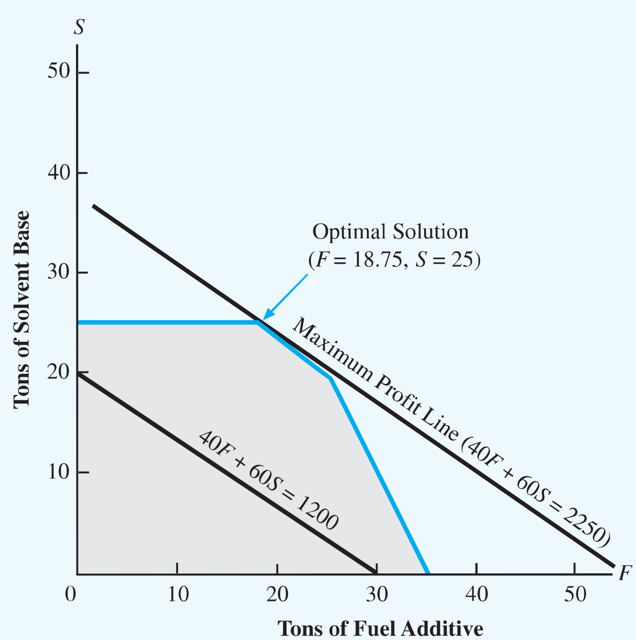
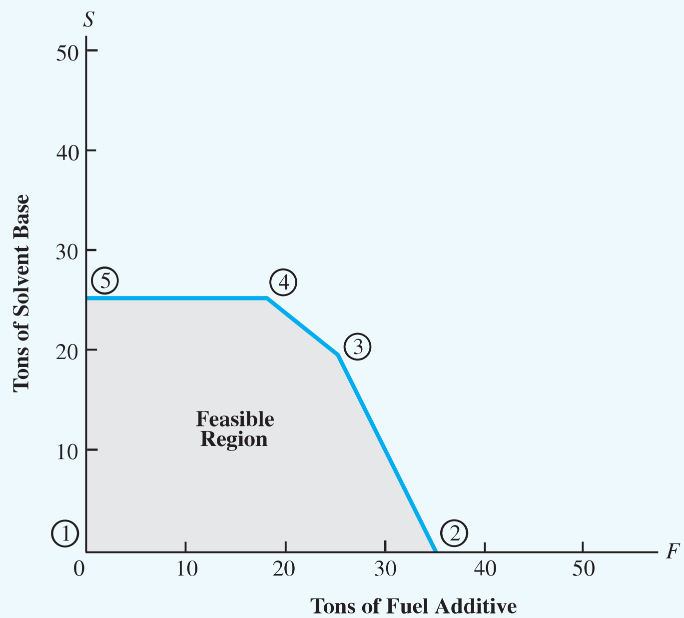


FIGURE 7.14 THE FIVE EXTREME POINTS OF THE FEASIBLE REGION FOR THE RMC PROBLEM



the RMC has five vertices or five extreme points (Figure 7.14). We can now state our observation about the location of optimal solutions:[2]

> The optimal solution to a linear programming problem can be found at an extreme point of the feasible region for the problem.

For additional practice in identifying the extreme points of the feasible region and determining the optimal solution by computing and comparing the objective function value at each extreme point, try Problem 13.

This property means that, if you are looking for the optimal solution to a linear programming problem, you do not have to evaluate all feasible solution points. In fact, you have to consider *only* the feasible solutions that occur at the extreme points of the feasible region. Thus, for the RMC problem, instead of computing and comparing the profit for all feasible solutions, we can find the optimal solution by evaluating the five extreme-point solutions and selecting the one that provides the highest profit. Actually, the graphical solution procedure is nothing more than a convenient way of identifying an optimal extreme point for two-variable problems.

7.4 Computer Solution of the RMC Problem

Computer programs designed to solve linear programming problems are widely available. After a short period of familiarization with the specific features of the program, most users can solve linear programming problems with few difficulties. Problems involving thousands of variables and thousands of constraints are now routinely solved with computer packages. Some of the leading commercial packages include CPLEX, LINGO, MOSEK, Gurobi, and Xpress-MP.

[2]In Section 7.6 we show that two special cases (infeasibility and unboundedness) in linear programming have no optimal solution. The observation stated does not apply to these cases.

In January 1952 the first successful computer solution of a linear programming problem was performed on the SEAC (Standards Eastern Automatic Computer). The SEAC, the first digital computer built by the National Bureau of Standards under U.S. Air Force sponsorship, had a 512-word memory and magnetic tape for external storage.

Instructions on how to solve linear programs using Excel and LINGO are provided in appendixes at the end of the chapter.

Packages are also available for free download. A good example is Clp (COIN-OR linear programming) available from the COIN-OR organization at *http://www.coin-or.org*.

A large number of user-friendly computer programs that can solve linear programs are now available. These programs, developed by academicians and small software companies, are almost all easy to use. Most of these programs are designed to solve smaller linear programs (a few hundred variables), but some can be used to solve problems involving thousands of variables and constraints. Linear programming solvers are also available in the spreadsheet environment. In Appendix 7.1 we show how to use the solver available with Excel. Appendix 7.2 demonstrates the use of LINGO, a stand-alone software package for solving optimization problems.

Probably the most widely used tool is Solver, which is built into Microsoft Excel. Therefore, the computer output we discuss is based on the output provided by Excel Solver. The complete details for how to formulate the RMC problem in Excel and use Solver are contained in Appendix 7.1.

Recall the RMC linear program:

$$
\begin{aligned}
\text{Max} \quad & 40F + 30S \\
\text{s.t.} \quad & \\
& 0.4F + 0.5S \leq 20 \quad \text{Material 1} \\
& \qquad\quad 0.2S \leq 5 \quad \text{Material 2} \\
& 0.6F + 0.3S \leq 21 \quad \text{Material 3} \\
& F, S \geq 0
\end{aligned}
$$

Figure 7.15 shows the optimal solution to the RMC problem. This output is based on the Answer Report from Excel Solver, but includes the variable names we have used in our linear programming model. This allows you to easily link the answer report to the model under discussion. We will use this style to show the solutions to optimization problems throughout Chapters 7–12.

Interpretation of Answer Report

Let us look more closely at the answer report in Figure 7.15 and interpret the computer solution provided for the RMC problem. First, note the number 1600.000 in the Objective Cells

FIGURE 7.15 ANSWER REPORT FOR THE RMC PROBLEM

WEB file RMC

Objective Cells (Max)

Name	Original Value	Final Value
Maximize Total Profit	0.000	1600.000

Variable Cells

Model Variable	Name	Original Value	Final Value	Integer
F	Tons Produced Fuel Additive	0.000	25.000	Contin
S	Tons Produced Solvent Base	0.000	20.000	Contin

Constraints

Constraint Number	Name	Cell Value	Status	Slack
1	Material 1 Amount Used	20.000	Binding	0.000
2	Material 2 Amount Used	4.000	Not Binding	1.000
3	Material 3 Amount Used	21.000	Binding	0.000

(Max) section, which appears in the Final Value column to the right of objective function value, Maximize Total Profit. This number indicates that the optimal solution to this problem will provide a profit of $1600. Directly below the objective function value are the values of the decision variables at the optimal solution. These are shown as the Final Value column of the Variable Cells section in the answer report. Thus, we have $F = 25$ tons of fuel additive and $S = 20$ tons of solvent base as the optimal production quantities. We will discuss the meaning of the Integer column in the Variable Cells section in Chapter 11.

The Constraints section of the answer report provides information about the status of the constraints. Recall that the RMC problem had three less-than-or-equal-to constraints corresponding to the tons available for each of the three raw materials. The information shown in the Slack column provides the value of the slack variable for each of the three constraints. This information is summarized as follows:

Constraint Number	Constraint Name	Value of Slack Variable
1	Material 1 Amount Used	0
2	Material 2 Amount Used	1
3	Material 3 Amount Used	0

Thus, we see that the binding constraints (the Material 1 Amount Used and Material 3 Amount Used constraints) have zero slack at the optimal solution. The Material 2 Amount Used constraint has 1 ton of slack, or unused capacity.

7.5 A Simple Minimization Problem

M&D Chemicals produces two products that are sold as raw materials to companies manufacturing bath soaps and laundry detergents. Based on an analysis of current inventory levels and potential demand for the coming month, M&D's management has specified that the combined production for products A and B must total at least 350 gallons. Separately, a major customer's order for 125 gallons of product A must also be satisfied. Product A requires 2 hours of processing time per gallon while product B requires 1 hour of processing time per gallon, and for the coming month, 600 hours of processing time are available. M&D's objective is to satisfy these requirements at a minimum total production cost. Production costs are $2 per gallon for product A and $3 per gallon for product B.

To find the minimum-cost production schedule, we will formulate the M&D Chemicals problem as a linear program. Following a procedure similar to the one used for the RMC problem, we first define the decision variables and the objective function for the problem. Let

$$A = \text{number of gallons of product A}$$
$$B = \text{number of gallons of product B}$$

Because the production costs are $2 per gallon for product A and $3 per gallon for product B, the objective function that corresponds to the minimization of the total production cost can be written as

$$\text{Min } 2A + 3B$$

Next, consider the constraints placed on the M&D Chemicals problem. To satisfy the major customer's demand for 125 gallons of product A, we know A must be at least 125. Thus, we write the constraint

$$1A \geq 125$$

Because the combined production for both products must total at least 350 gallons, we can write the constraint

$$1A + 1B \geq 350$$

Finally, the limitation on available processing time of 600 hours means that we need to add the constraint

$$2A + 1B \leq 600$$

After adding the nonnegativity constraints ($A, B \geq 0$), we have the following linear program for the M&D Chemicals problem:

$$\begin{aligned} \text{Max} \quad & 2A + 3B \\ \text{s.t.} \quad & \\ & 1A \geq 125 && \text{Demand for product A} \\ & 1A + 1B \geq 350 && \text{Total production} \\ & 2A + 1B \leq 600 && \text{Processing time} \\ & A, B \geq 0 \end{aligned}$$

Because the linear programming model has only two decision variables, the graphical solution procedure can be used to find the optimal production quantities. The graphical method for this problem, just as in the RMC problem, requires us to first graph the constraint lines to find the feasible region. By graphing each constraint line separately and then checking points on either side of the constraint line, the solutions that satisfy each constraint can be identified. By combining the solutions that satisfy each constraint on the same graph, we obtain the feasible region shown in Figure 7.16.

FIGURE 7.16 FEASIBLE REGION FOR THE M&D CHEMICALS PROBLEM

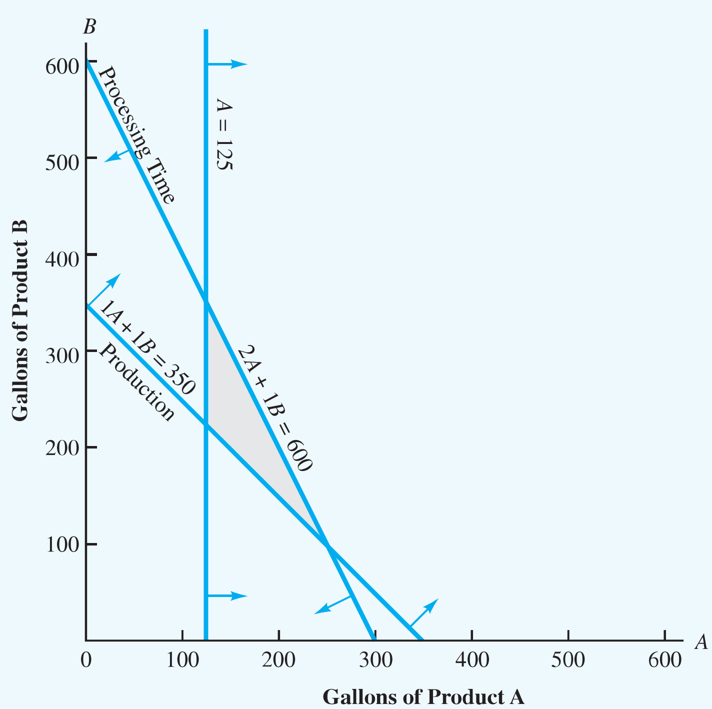
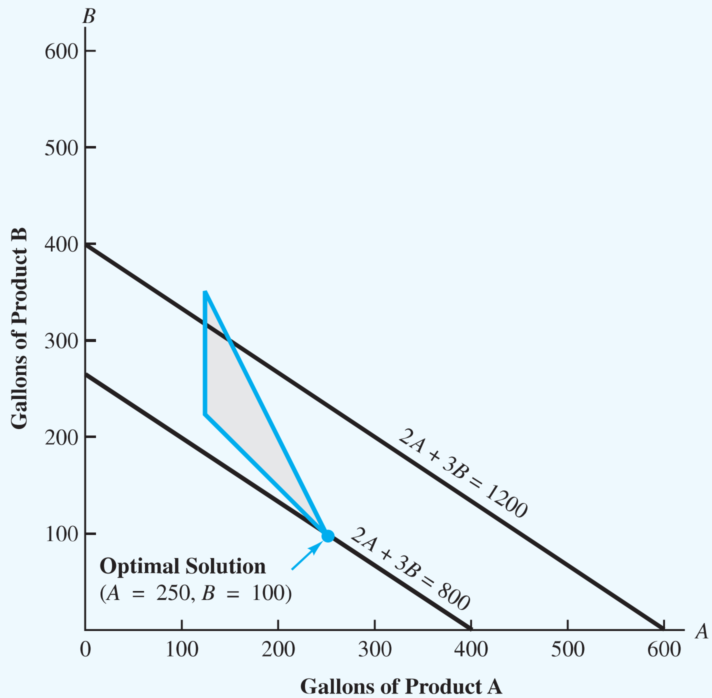


FIGURE 7.17 GRAPHICAL SOLUTION FOR THE M&D CHEMICALS PROBLEM



To find the minimum-cost solution, we now draw the objective function line corresponding to a particular total cost value. For example, we might start by drawing the line $2A + 3B = 1200$. This line is shown in Figure 7.17. Clearly some points in the feasible region would provide a total cost of \$1200. To find the values of A and B that provide smaller total cost values, we move the objective function line in a lower left direction until, if we moved it any farther, it would be entirely outside the feasible region. Note that the objective function line $2A + 3B = 800$ intersects the feasible region at the extreme point $A = 250$ and $B = 100$. This extreme point provides the minimum-cost solution with an objective function value of 800. From Figures 7.16 and 7.17, we can see that the total production constraint and the processing time constraint are binding. Just as in every linear programming problem, the optimal solution occurs at an extreme point of the feasible region.

Summary of the Graphical Solution Procedure for Minimization Problems

Can you use the graphical solution procedure to determine the optimal solution for a minimization problem? Try Problem 31.

The steps of the graphical solution procedure for a minimization problem are summarized here:

1. Prepare a graph for each constraint that shows the solutions that satisfy the constraint.
2. Determine the feasible region by identifying the solutions that satisfy all the constraints simultaneously.

3. Draw an objective function line showing the values of the decision variables that yield a specified value of the objective function.
4. Move parallel objective function lines toward smaller objective function values until further movement would take the line completely outside the feasible region.
5. Any feasible solution on the objective function line with the smallest value is an optimal solution.

Surplus Variables

The optimal solution to the M&D Chemicals problem shows that the desired total production of $A + B = 350$ gallons is achieved by using all available processing time of $2A + 1B = 2(250) + 1(100) = 600$ hours. In addition, note that the constraint requiring that product A demand be met is satisfied with $A = 250$ gallons. In fact, the production of product A exceeds its minimum level by $250 - 125 = 125$ gallons. This excess production for product A is referred to as *surplus.* In linear programming terminology, any excess quantity corresponding to a $\geq$ constraint is referred to as surplus.

Excel Solver refers to all nonbinding constraints as having positive slack values regardless of whether they are $\geq$ or $\leq$ constraints. However, we will use "surplus" when referring to a nonbinding $\geq$ constraint.

Recall that with a $\leq$ constraint, a slack variable can be added to the left-hand side of the inequality to convert the constraint to equality form. With a $\geq$ constraint, a **surplus variable** can be subtracted from the left-hand side of the inequality to convert the constraint to equality form. Just as with slack variables, surplus variables are given a coefficient of zero in the objective function because they have no effect on its value. After including two surplus variables, S_1 and S_2, for the $\geq$ constraints and one slack variable, S_3, for the $\leq$ constraint, the linear programming model of the M&D Chemicals problem becomes

$$\begin{aligned} \text{Min} \quad & 2A + 3B + 0S_1 + 0S_2 + 0S_3 \\ \text{s.t.} \quad & \\ & 1A - 1S_1 = 125 \\ & 1A + 1B - 1S_2 = 350 \\ & 2A + 1B + 1S_3 = 600 \\ & A, B, S_1, S_2, S_3 \geq 0 \end{aligned}$$

Try Problem 35 to test your ability to use slack and surplus variables to write a linear program in standard form.

All the constraints are now equalities. Hence, the preceding formulation is the standard form representation of the M&D Chemicals problem. At the optimal solution of $A = 250$ and $B = 100$, the values of the surplus and slack variables are as follows:

Constraint	Value of Surplus or Slack Variable
Demand for product A	$S_1 = 125$
Total production	$S_2 = 0$
Processing time	$S_3 = 0$

Refer to Figures 7.16 and 7.17. Note that the zero surplus and slack variables are associated with the constraints that are binding at the optimal solution—that is, the total production and processing time constraints. The surplus of 125 units is associated with the nonbinding constraint on the demand for product A.

In the RMC problem all the constraints were of the $\leq$ type, and in the M&D Chemicals problem the constraints were a mixture of $\geq$ and $\leq$ types. The number and types of constraints encountered in a particular linear programming problem depend on the specific conditions existing in the problem. Linear programming problems may have some $\leq$ constraints, some $\geq$ constraints, and some $=$ constraints. For an equality constraint, feasible solutions must lie directly on the constraint line.

Try Problem 34 to practice solving a linear program with all three constraint forms.

An example of a linear program with two decision variables, G and H, and all three constraint forms is given here:

$$\begin{aligned} \text{Min} \quad & 2G + 2H \\ \text{s.t.} \quad & \\ & 1G + 3H \leq 12 \\ & 3G + 1H \geq 13 \\ & 1G - 1H = 3 \\ & G, H \geq 0 \end{aligned}$$

The standard-form representation of this problem is

$$\begin{aligned} \text{Min} \quad & 2G + 2H + 0S_1 + 0S_2 \\ \text{s.t.} \quad & \\ & 1G + 3H + 1S_1 = 12 \\ & 3G + 1H - 1S_2 = 13 \\ & 1G - 1H = 3 \\ & G, H, S_1, S_2 \geq 0 \end{aligned}$$

The standard form requires a slack variable for the $\leq$ constraint and a surplus variable for the $\geq$ constraint. However, neither a slack nor a surplus variable is required for the third constraint because it is already in equality form.

When solving linear programs graphically, it is not necessary to write the problem in its standard form. Nevertheless, it is helpful to be able to compute the values of the slack and surplus variables and understand what they mean. A final point: The standard form of the linear programming problem is equivalent to the original formulation of the problem. That is, the optimal solution to any linear programming problem is the same as the optimal solution to the standard form of the problem. The standard form does not change the basic problem; it only changes how we write the constraints for the problem.

Computer Solution of the M&D Chemicals Problem

The answer report for the M&D Chemicals Problem is presented in Figure 7.18. The answer report shows that the minimum-cost solution yields an objective function value of $800. The

FIGURE 7.18 ANSWER REPORT FOR THE M&D CHEMICALS PROBLEM

WEB file
M&D

Objective Cells (Min)

Name	Original Value	Final Value
Minimize Total Cost Product A	0.000	800.000

Variable Cells

Model Variable	Name	Original Value	Final Value	Integer
A	Gallons Produced Product A	0.000	250.000	Contin
B	Gallons Produced Product B	0.000	100.000	Contin

Constraints

Constraint Number	Name	Cell Value	Status	Slack
1	Demand for Product A	250.000	Not Binding	125.000
2	Total Production	350.000	Binding	0.000
3	Processing Time	600.000	Binding	0.000

values of the decision variables show that 250 gallons of product A and 100 gallons of product B provide the minimum-cost solution.

The Slack column in the Constraints section of the answer report shows that the $\geq$ constraint corresponding to the demand for product A (see constraint 1) has a value of 125 units. Excel uses "slack" when referring to nonbinding $\geq$ or $\leq$ constraints. However, since this is a $\geq$ constraint, it tells us that production of product A in the optimal solution exceeds demand by 125 gallons. In other words, the demand for product A (constraint 1) has a surplus value of 125 units. The slack values are zero for the total production requirement (constraint 2) and the processing time limitation (constraint 3), which indicates that these constraints are binding at the optimal solution.

7.6 Special Cases

In this section we discuss three special situations that can arise when we attempt to solve linear programming problems.

Alternative Optimal Solutions

From our discussion of the graphical solution procedure, we know that optimal solutions can be found at the extreme points of the feasible region. Now let us consider the special case where the optimal objective function line coincides with one of the binding constraint lines. It can lead to **alternative optimal solutions**, whereby more than one solution provides the optimal value for the objective function.

To illustrate the case of alternative optimal solutions, we return to the RMC problem. However, let us assume that the profit contribution for the solvent base (S) has increased to \$50. The revised objective function is $40F + 50S$. Figure 7.19 shows the graphical solution to this problem. Note that the optimal solution still occurs at an extreme point. In fact, it occurs at two extreme points: extreme point ③ ($F = 25$, $S = 20$) and extreme point ④ ($F = 18.75$, $S = 25$).

The objective function values at these two extreme points are identical; that is,

$$40F + 50S = 40(25) + 50(20) = 2000$$

and

$$40F + 50S = 40(18.75) + 50(25) = 2000$$

Furthermore, any point on the line connecting the two optimal extreme points also provides an optimal solution. For example, the solution point ($F = 21.875$, $S = 22.5$), which is halfway between the two extreme points, also provides the optimal objective function value of

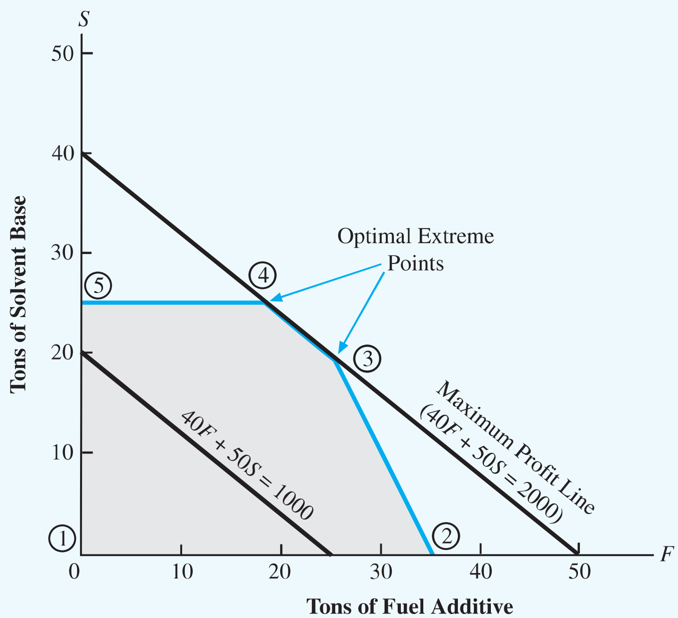
$$40F + 50S = 40(21.875) + 50(22.5) = 2000$$

A linear programming problem with alternative optimal solutions is generally a good situation for the manager or decision maker. It means that several combinations of the decision variables are optimal and that the manager can select the most desirable optimal solution. Unfortunately, determining whether a problem has alternative optimal solutions is not a simple matter.

Infeasibility

Infeasibility means that no solution to the linear programming problem satisfies all constraints, including the nonnegativity constraints. Graphically, infeasibility means that a feasible region does not exist; that is, no points satisfy all constraint equations and

FIGURE 7.19 OPTIMAL SOLUTIONS FOR THE RMC PROBLEM WITH AN OBJECTIVE FUNCTION OF $40F + 50S$



nonnegativity conditions simultaneously. To illustrate this situation, let us return to the problem facing RMC.

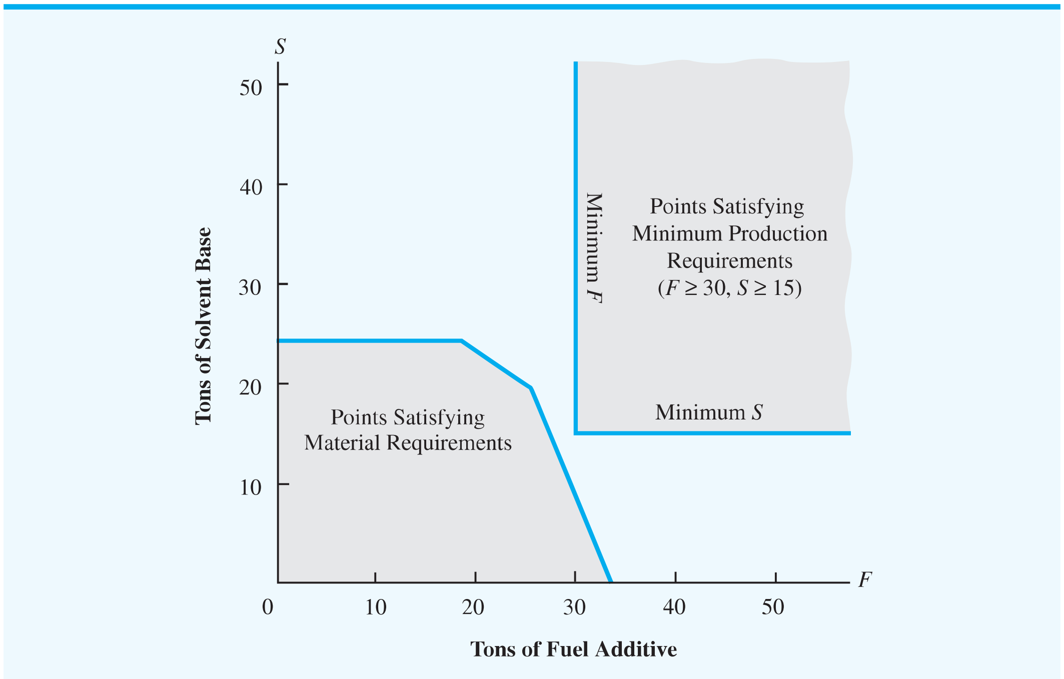
Problems with no feasible solution do arise in practice, most often because management's expectations are too high or because too many restrictions have been placed on the problem.

Suppose that management specified that at least 30 tons of fuel additive and at least 15 tons of solvent base must be produced. Figure 7.20 shows the graph of the solution region that reflects these requirements. The shaded area in the lower left-hand portion of the graph depicts those points satisfying the less-than-or-equal-to constraints on the amount of materials available. The shaded area in the upper right-hand portion depicts those points satisfying the minimum production requirements of 30 tons of fuel additive and 15 tons of solvent base. But none of the points satisfy both sets of constraints. Thus, if management imposes these minimum production requirements, no feasible solution to the linear programming problem is possible.

How should we interpret this infeasibility in terms of the current problem? First, we should tell management that, for the available amounts of the three materials, producing 30 tons of fuel additive and 15 tons of solvent base isn't possible. Moreover, we can tell management exactly how much more of each material is needed.

Material	Minimum Tons Required for $F = 30, S = 15$	Tons Available	Additional Tons Required
Material 1	$0.4(30) + 0.5(15) = 19.5$	20	—
Material 2	$0.2(15) = 3$	5	—
Material 3	$0.6(30) + 0.3(15) = 22.5$	21	1.5

FIGURE 7.20 NO FEASIBLE REGION FOR THE RMC PROBLEM WITH MINIMUM PRODUCTION REQUIREMENTS



Thus, RMC has a sufficient supply of materials 1 and 2 but will need 1.5 additional tons of material 3 to meet management's production requirements of 30 tons of fuel additive and 15 tons of solvent base. If, after reviewing the preceding analysis, management still wants this level of production for the two products, RMC will have to obtain the additional 1.5 tons of material 3.

Often, many possibilities are available for corrective management action, once we discover the lack of a feasible solution. The important thing to realize is that linear programming analysis can help determine whether management's plans are feasible. By analyzing the problem using linear programming, we are often able to point out infeasible conditions and initiate corrective action.

Whenever you attempt to solve a problem that is infeasible using Excel Solver, you will obtain a message that says "Solver could not find a feasible solution." In this case, you know that no solution to the linear programming problem will satisfy all constraints. Careful inspection of your formulation is necessary to identify why the problem is infeasible. In some situations the only reasonable approach is to drop one or more constraints and resolve the problem. If you are able to find an optimal solution for this revised problem, you will know that the constraint(s) that were omitted are causing the problem to be infeasible.

Unbounded

The solution to a maximization linear programming problem is **unbounded** if the value of the solution may be made infinitely large without violating any of the constraints; for a minimization problem, the solution is unbounded if the value may be made infinitely small. This condition might be termed *managerial utopia*; for example, if this condition were to occur in a profit maximization problem, the manager could achieve an unlimited profit.

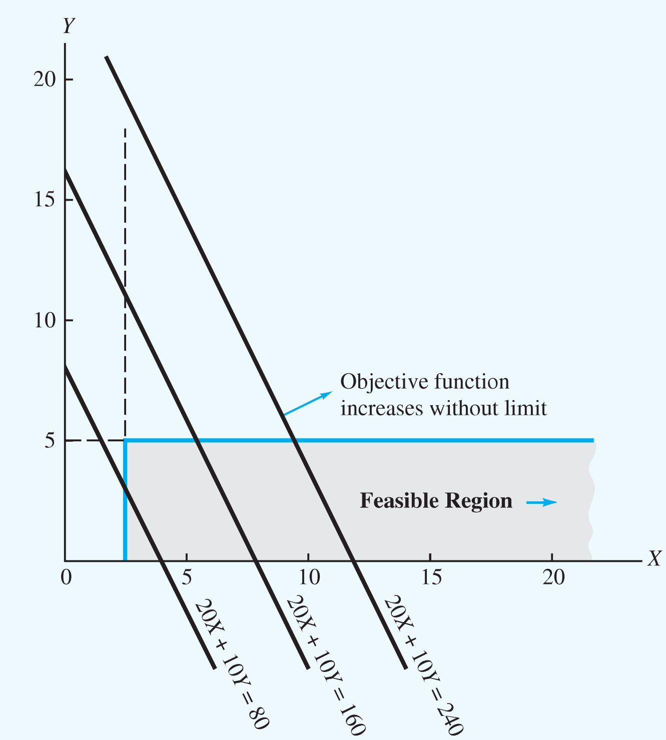
However, in linear programming models of real problems, the occurrence of an unbounded solution means that the problem has been improperly formulated. We know it is not possible to increase profits indefinitely. Therefore, we must conclude that if a profit maximization problem results in an unbounded solution, the mathematical model doesn't represent the real-world problem sufficiently. Usually, an unbounded problem results from the inadvertent omission of a constraint during problem formulation.

As an illustration, consider the following linear program with two decision variables, *X* and *Y*:

$$\begin{aligned} \text{Max} \quad & 20X + 10Y \\ \text{s.t.} \quad & \\ & 1X \qquad\quad \geq 2 \\ & \qquad\quad 1Y \leq 5 \\ & X, Y \geq 0 \end{aligned}$$

In Figure 7.21 we graphed the feasible region associated with this problem. Note that we can only indicate part of the feasible region because the feasible region extends indefinitely in the direction of the *X*-axis. Looking at the objective function lines in Figure 7.21, we see that the solution to this problem may be made as large as we desire. No matter what solution we pick, we will always be able to reach some feasible solution with a larger value. Thus, we say that the solution to this linear program is *unbounded.*

FIGURE 7.21 EXAMPLE OF AN UNBOUNDED PROBLEM



Can you recognize whether a linear program involves alternative optimal solutions, infeasibility, or is unbounded? Try Problems 42 and 43.

Whenever you attempt to solve a problem that is unbounded using Excel Solver, you will obtain a message that says, "The Objective Cell values do not converge." Because unbounded solutions cannot occur in real problems, the first thing you should do is to review your model to determine whether you have incorrectly formulated the problem.

NOTES AND COMMENTS

1. Infeasibility is independent of the objective function. It exists because the constraints are so restrictive that they allow no feasible region for the linear programming model. Thus, when you encounter infeasibility, making changes in the coefficients of the objective function will not help; the problem will remain infeasible.
2. The occurrence of an unbounded solution is often the result of a missing constraint. However, a change in the objective function may cause a previously unbounded problem to become bounded with an optimal solution. For example, the graph in Figure 7.21 shows an unbounded solution for the objective function Max $20X + 10Y$. However, changing the objective function to Max $-20X - 10Y$ will provide the optimal solution $X = 2$ and $Y = 0$ even though no changes have been made in the constraints.

7.7 General Linear Programming Notation

In this chapter we showed how to formulate mathematical models for the RMC and M&D Chemicals linear programming problems. To formulate a mathematical model of the RMC problem, we began by defining two decision variables: F = number of tons of fuel additive, and S = number of tons of solvent base. In the M&D Chemicals problem, the two decision variables were defined as A = number of gallons of product A, and B = number of gallons of product B. We selected decision variable names of F and S in the RMC problem and A and B in the M&D Chemicals problem to make it easier to recall what these decision variables represented in the problem. Although this approach works well for linear programs involving a small number of decision variables, it can become difficult when dealing with problems involving a large number of decision variables.

A more general notation that is often used for linear programs uses the letter x with a subscript. For instance, in the RMC problem, we could have defined the decision variables as follows:

$$x_1 = \text{number of tons of fuel additive}$$
$$x_2 = \text{number of tons of solvent base}$$

In the M&D Chemicals problem, the same variable names would be used, but their definitions would change:

$$x_1 = \text{number of gallons of product A}$$
$$x_2 = \text{number of gallons of product B}$$

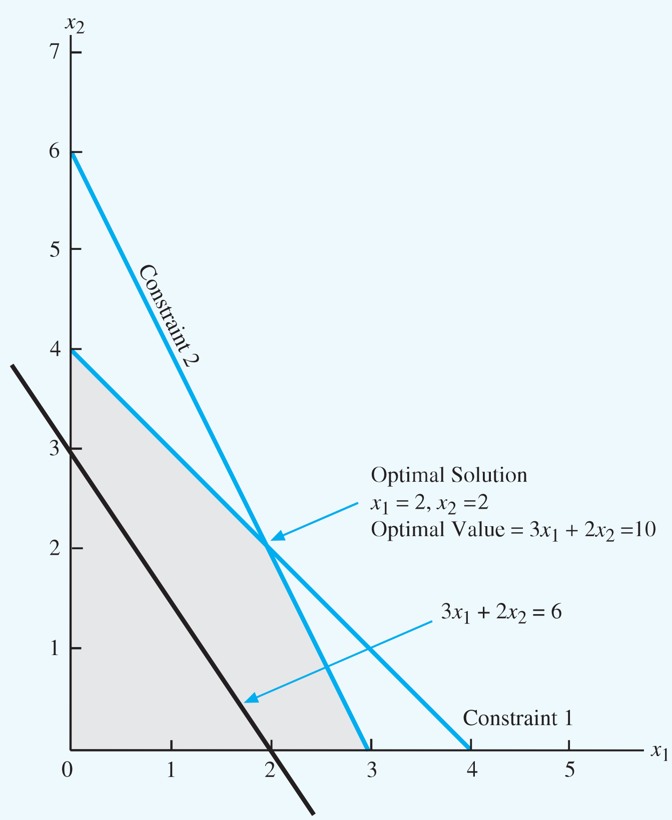
A disadvantage of using general notation for decision variables is that we are no longer able to easily identify what the decision variables actually represent in the mathematical model. However, the advantage of general notation is that formulating a mathematical model for a problem that involves a large number of decision variables is much easier. For instance, for a linear programming problem with three decision variables, we would use variable names of x_1, x_2, and x_3; for a problem with four decision variables, we would use variable names of x_1, x_2, x_3, and x_4; and so on. Clearly, if a problem involved 1000 decision variables, trying to identify 1000 unique names would be difficult. However, using the general linear programming notation, the decision variables would be defined as $x_1, x_2, x_3, \ldots, x_{1000}$.

To illustrate the graphical solution procedure for a linear program written using general linear programming notation, consider the following mathematical model for a maximization problem involving two decision variables:

$$\begin{aligned} \text{Max} \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & \\ & 2x_1 + 2x_2 \le 8 \\ & 1x_1 + 0.5x_2 \le 3 \\ & x_1, x_2 \ge 0 \end{aligned}$$

We must first develop a graph that displays the possible solutions (x_1 and x_2 values) for the problem. The usual convention is to plot values of x_1 along the horizontal axis and values of x_2 along the vertical axis. Figure 7.22 shows the graphical solution for this two-variable

FIGURE 7.22 GRAPHICAL SOLUTION OF A TWO-VARIABLE LINEAR PROGRAM WITH GENERAL NOTATION



problem. Note that for this problem the optimal solution is $x_1 = 2$ and $x_2 = 2$, with an objective function value of 10.

Using general linear programming notation, we can write the standard form of the preceding problem as follows:

$$
\begin{aligned}
\text{Max} \quad & 3x_1 + 2x_2 + 0s_1 + 0s_2 \\
\text{s.t.} \quad & \\
& 2x_1 + 2x_2 + 1s_1 = 8 \\
& 1x_1 + 0.5x_2 + 1s_2 = 3 \\
& x_1, x_2, s_1, s_2 \geq 0
\end{aligned}
$$

Thus, at the optimal solution $x_1 = 2$ and $x_2 = 2$, the values of the slack variables are $s_1 = s_2 = 0$.

Summary

We formulated linear programming models for the RMC maximization problem and the M&D Chemicals minimization problem. For both problems we showed how a graphical solution procedure and Excel Solver can be used to identify an optimal solution. In formulating a linear programming model of these problems, we developed a general definition of a linear program.

A linear program is a mathematical model with the following qualities:

1. A linear objective function that is to be maximized or minimized
2. A set of linear constraints
3. Variables restricted to nonnegative values

Slack variables may be used to write less-than-or-equal-to constraints in equality form, and surplus variables may be used to write greater-than-or-equal-to constraints in equality form. The value of a slack variable can usually be interpreted as the amount of unused resource, whereas the value of a surplus variable indicates the amount over and above some stated minimum requirement. When all constraints have been written as equalities, the linear program has been written in its standard form.

If the solution to a linear program is infeasible or unbounded, no optimal solution to the problem can be found. In the case of infeasibility, no feasible solutions are possible. In the case of an unbounded solution, the objective function can be made infinitely large for a maximization problem and infinitely small for a minimization problem. In the case of alternative optimal solutions, two or more optimal extreme points exist, and all the points on the line segment connecting them are also optimal.

The chapter concluded with a section showing how to write a mathematical model using general linear programming notation. The Q.M. in Action, The U.S. Navy Uses Linear Programming for Fuel Conservation, provides just one of many examples of the widespread use of linear programming. In the next two chapters we will see many more applications of linear programming.

Q.M. *in* ACTION

*THE U.S. NAVY USES LINEAR PROGRAMMING FOR FUEL CONSERVATION**

The U.S. Navy spends approximately one billion dollars per year on fuel for its ships. The major determinant of fuel consumption for a ship is speed. A ship can be operated in a number of different modes. For example, a naval destroyer ship, with four engines available to run two shafts for propelling the ship, may operate in three different modes: trail-shaft, split-plant, and full-power. In trail-shaft mode, one of the four engines runs a single shaft. In split-plant mode, two engines run, each driving its own shaft. Full-power mode has all four engines running, two on each shaft. Full-power has the highest full consumption rate as measured in gallons per hour. Depending on the mode, a ship may consume between 600 to 7000 gallons of fuel per hour.

The captain of a ship which needs to get from point A to point B in a certain amount of time has mode options, each of which consumes different amounts of fuel. The ship can run at a steady speed or at variable speeds. For example, a ship can operate in a faster mode during first portion of the trip and in a slower mode during the remaining portion of the trip and still arrive at the same time as a vessel running at the required constant speed. The U.S. Navy uses a spreadsheet-based linear programming model to determine the most fuel efficient way to make a trip in a required amount of time.

The decision variables of the linear program are the amount of time (in hours) to run at a given speed. The objective is to minimize fuel consumption subject to constraints that the required distance must be covered within a specified amount of time. The solution to the linear program specifies the amount of time to run at each speed, but the order in which the speeds are implemented is flexible. Compared to the fuel consumption when operating in a single mode, fuel savings ranging from 2% to 54% have been realized by implementing the variable modes recommended by the linear programming model.

*Based on G. Brown, J. Kline, R. Rosenthal, and A. Washburn, "Steaming on Convex Hulls," *Interfaces* 37, no. 4 (July–August 2007): 342–352.

Glossary

Constraint An equation or inequality that rules out certain combinations of decision variables as feasible solutions.

Problem formulation The process of translating a verbal statement of a problem into a mathematical statement called the *mathematical model.*

Mathematical model A representation of a problem where the objective and all constraint conditions are described by mathematical expressions.

Decision variable A controllable input for a linear programming model.

Objective function The expression that defines the quantity to be maximized or minimized in a linear programming model.

Nonnegativity constraints A set of constraints that requires all variables to be nonnegative.

Linear program A mathematical model with a linear objective function, a set of linear constraints, and nonnegative variables.

Linear functions Mathematical expressions in which the variables appear in separate terms and are raised to the first power.

Feasible solution A solution that satisfies all the constraints simultaneously.

Feasible region The set of all feasible solutions.

Slack variable A variable added to the left-hand side of a less-than-or-equal-to constraint to convert the constraint into an equality. The value of this variable can usually be interpreted as the amount of unused resource.

Standard form A linear program in which all the constraints are written as equalities. The optimal solution of the standard form of a linear program is the same as the optimal solution of the original formulation of the linear program.

Redundant constraint A constraint that does not affect the feasible region. If a constraint is redundant, it can be removed from the problem without affecting the feasible region.

Extreme point Graphically speaking, extreme points are the feasible solution points occurring at the vertices, or "corners," of the feasible region. With two-variable problems, extreme points are determined by the intersection of the constraint lines.

Surplus variable A variable subtracted from the left-hand side of a greater-than-or-equal-to constraint to convert the constraint into an equality. The value of this variable can usually be interpreted as the amount over and above some required minimum level.

Alternative optimal solutions The case in which more than one solution provides the optimal value for the objective function.

Infeasibility The situation in which no solution to the linear programming problem satisfies all the constraints.

Unbounded The situation in which the value of the solution may be made infinitely large in a maximization linear programming problem or infinitely small in a minimization problem without violating any of the constraints.

Problems

SELF test

1. Which of the following mathematical relationships could be found in a linear programming model, and which could not? For the relationships that are unacceptable for linear programs, state why.
   a. $-1A + 2B \leq 70$
   b. $2A - 2B = 50$
   c. $1A - 2B^2 \leq 10$
   d. $3\sqrt{A} + 2B \geq 15$
   e. $1A + 1B = 6$
   f. $2A + 5B + 1AB \leq 25$

SELF test

2. Find the solutions that satisfy the following constraints:
   a. $4A + 2B \leq 16$
   b. $4A + 2B \geq 16$
   c. $4A + 2B = 16$
3. Show a separate graph of the constraint lines and the solutions that satisfy each of the following constraints:
   a. $3A + 2B \leq 18$
   b. $12A + 8B \geq 480$
   c. $5A + 10B = 200$
4. Show a separate graph of the constraint lines and the solutions that satisfy each of the following constraints:
   a. $3A - 4B \geq 60$
   b. $-6A + 5B \leq 60$
   c. $5A - 2B \leq 0$

5. Show a separate graph of the constraint lines and the solutions that satisfy each of the following constraints:
   a. $A \geq 0.25\,(A + B)$
   b. $B \leq 0.10\,(A + B)$
   c. $A \leq 0.50\,(A + B)$
6. Three objective functions for linear programming problems are $7A + 10B$, $6A + 4B$, and $-4A + 7B$. Show the graph of each for objective function values equal to 420.
7. Identify the feasible region for the following set of constraints:

$$\begin{aligned} 0.5A + 0.25B &\geq 30 \\ 1A + 5B &\geq 250 \\ 0.25A + 0.5B &\leq 50 \\ A, B &\geq 0 \end{aligned}$$

8. Identify the feasible region for the following set of constraints:

$$\begin{aligned} 2A - 1B &\leq 0 \\ -1A + 1.5B &\leq 200 \\ A, B &\geq 0 \end{aligned}$$

9. Identify the feasible region for the following set of constraints:

$$\begin{aligned} 3A - 2B &\geq 0 \\ 2A - 1B &\leq 200 \\ 1A &\leq 150 \\ A, B &\geq 0 \end{aligned}$$

10. For the linear program

$$\begin{aligned} \text{Max} \quad & 2A + 3B \\ \text{s.t.} \quad & \\ & 1A + 2B \leq 6 \\ & 5A + 3B \leq 15 \\ & A, B \geq 0 \end{aligned}$$

find the optimal solution using the graphical solution procedure. What is the value of the objective function at the optimal solution?

11. Solve the following linear program using the graphical solution procedure:

$$\begin{aligned} \text{Max} \quad & 5A + 5B \\ \text{s.t.} \quad & \\ & 1A \leq 100 \\ & 1B \leq 80 \\ & 2A + 4B \leq 400 \\ & A, B \geq 0 \end{aligned}$$

12. Consider the following linear programming problem:

$$\begin{aligned} \text{Max} \quad & 3A + 3B \\ \text{s.t.} \quad & \\ & 2A + 4B \leq 12 \\ & 6A + 4B \leq 24 \\ & A, B \geq 0 \end{aligned}$$

a. Find the optimal solution using the graphical solution procedure.
b. If the objective function is changed to $2A + 6B$, what will the optimal solution be?
c. How many extreme points are there? What are the values of A and B at each extreme point?

SELF test

13. Consider the following linear program:

$$\begin{aligned} \text{Max} \quad & 1A + 2B \\ \text{s.t.} \quad & \\ & 1A \leq 5 \\ & 1B \leq 4 \\ & 2A + 2B = 12 \\ & A, B \geq 0 \end{aligned}$$

a. Show the feasible region.
b. What are the extreme points of the feasible region?
c. Find the optimal solution using the graphical procedure.

14. Par, Inc., is a small manufacturer of golf equipment and supplies. Par's distributor believes a market exists for both a medium-priced golf bag, referred to as a standard model, and a high-priced golf bag, referred to as a deluxe model. The distributor is so confident of the market that, if Par can make the bags at a competitive price, the distributor will purchase all the bags that Par can manufacture over the next three months. A careful analysis of the manufacturing requirements resulted in the following table, which shows the production time requirements for the four required manufacturing operations and the accounting department's estimate of the profit contribution per bag:

	Production Time (hours)				
Product	Cutting and Dyeing	Sewing	Finishing	Inspection and Packaging	Profit per Bag
Standard	$^7/_{10}$	$^1/_2$	1	$^1/_{10}$	\$10
Deluxe	1	$^5/_6$	$^2/_3$	$^1/_4$	\$ 9

The director of manufacturing estimates that 630 hours of cutting and dyeing time, 600 hours of sewing time, 708 hours of finishing time, and 135 hours of inspection and packaging time will be available for the production of golf bags during the next three months.

a. If the company wants to maximize total profit contribution, how many bags of each model should it manufacture?
b. What profit contribution can Par earn on those production quantities?
c. How many hours of production time will be scheduled for each operation?
d. What is the slack time in each operation?

15. Suppose that Par's management (Problem 14) encounters the following situations:
a. The accounting department revises its estimate of the profit contribution for the deluxe bag to \$18 per bag.
b. A new low-cost material is available for the standard bag, and the profit contribution per standard bag can be increased to \$20 per bag. (Assume that the profit contribution of the deluxe bag is the original \$9 value.)
c. New sewing equipment is available that would increase the sewing operation capacity to 750 hours. (Assume that $10A + 9B$ is the appropriate objective function.)

If each of these situations is encountered separately, what is the optimal solution and the total profit contribution?

16. Refer to the feasible region for Par, Inc., in Problem 14.
    a. Develop an objective function that will make extreme point (0, 540) the optimal extreme point.
    b. What is the optimal solution for the objective function you selected in part (a)?
    c. What are the values of the slack variables associated with this solution?

SELF test

17. Write the following linear program in standard form:

$$
\begin{aligned}
\text{Max} \quad & 5A + 2B \\
\text{s.t.} \quad & \\
& 1A - 2B \le 420 \\
& 2A + 3B \le 610 \\
& 6A - 1B \le 125 \\
& A, B \ge 0
\end{aligned}
$$

18. For the linear program

$$
\begin{aligned}
\text{Max} \quad & 4A + 1B \\
\text{s.t.} \quad & \\
& 10A + 2B \le 30 \\
& 3A + 2B \le 12 \\
& 2A + 2B \le 10 \\
& A, B \ge 0
\end{aligned}
$$

    a. Write this problem in standard form.
    b. Solve the problem using the graphical solution procedure.
    c. What are the values of the three slack variables at the optimal solution?

19. Given the linear program

$$
\begin{aligned}
\text{Max} \quad & 3A + 4B \\
\text{s.t.} \quad & \\
& -1A + 2B \le 8 \\
& 1A + 2B \le 12 \\
& 2A + 1B \le 16 \\
& A, B \ge 0
\end{aligned}
$$

    a. Write the problem in standard form.
    b. Solve the problem using the graphical solution procedure.
    c. What are the values of the three slack variables at the optimal solution?

20. For the linear program

$$
\begin{aligned}
\text{Max} \quad & 3A + 2B \\
\text{s.t.} \quad & \\
& A + B \ge 4 \\
& 3A + 4B \le 24 \\
& A \ge 2 \\
& A - B \le 0 \\
& A, B \ge 0
\end{aligned}
$$

a. Write the problem in standard form.
b. Solve the problem.
c. What are the values of the slack and surplus variables at the optimal solution?

21. Consider the following linear program:

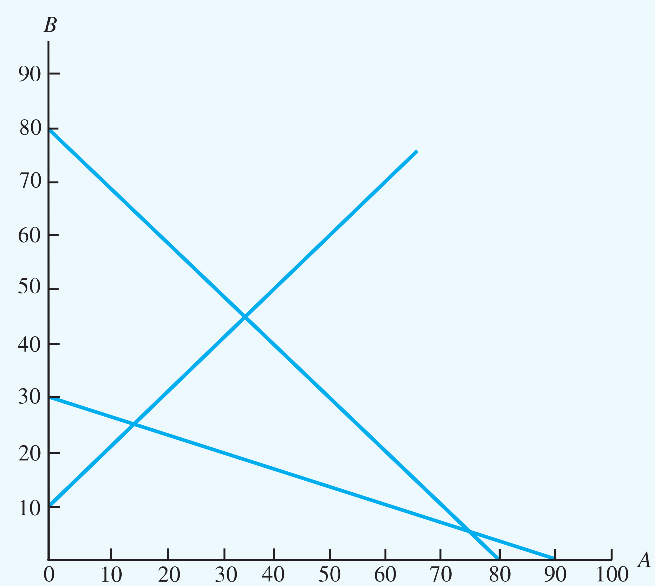
$$\begin{aligned} \text{Max} \quad & 2A + 3B \\ \text{s.t.} \quad & \\ & 5A + 5B \leq 400 \quad \text{Constraint 1} \\ & -1A + 1B \leq 10 \quad \text{Constraint 2} \\ & 1A + 3B \geq 90 \quad \text{Constraint 3} \\ & A, B \geq 0 \end{aligned}$$

Figure 7.23 shows a graph of the constraint lines.

a. Place a number (1, 2, or 3) next to each constraint line to identify which constraint it represents.
b. Shade in the feasible region on the graph.
c. Identify the optimal extreme point. What is the optimal solution?
d. Which constraints are binding? Explain.
e. How much slack or surplus is associated with the nonbinding constraint?

22. Reiser Sports Products wants to determine the number of All-Pro (A) and College (C) footballs to produce in order to maximize profit over the next four-week planning horizon. Constraints affecting the production quantities are the production capacities in three departments: cutting and dyeing; sewing; and inspection and packaging. For the four-week planning period, 340 hours of cutting and dyeing time, 420 hours of sewing time, and 200 hours of inspection and packaging time are available. All-Pro footballs provide a profit

FIGURE 7.23 GRAPH OF THE CONSTRAINT LINES FOR EXERCISE 21



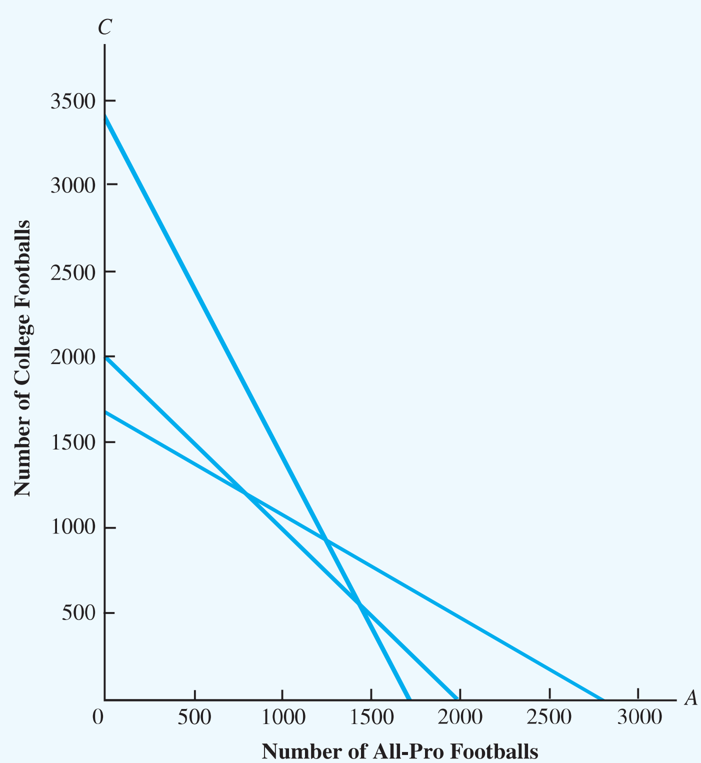
of \$5 per unit and College footballs provide a profit of \$4 per unit. The linear programming model with production times expressed in minutes is as follows:

$$
\begin{aligned}
\text{Max} \quad & 5A + 4C \\
\text{s.t.} \quad & \\
& 12A + 6C \le 20{,}400 \quad \text{Cutting and dyeing} \\
& 9A + 15C \le 25{,}200 \quad \text{Sewing} \\
& 6A + 6C \le 12{,}000 \quad \text{Inspection and packaging} \\
& A, C \ge 0
\end{aligned}
$$

A portion of the graphical solution to the Reiser problem is shown in Figure 7.24.

a. Shade the feasible region for this problem.
b. Determine the coordinates of each extreme point and the corresponding profit. Which extreme point generates the highest profit?
c. Draw the profit line corresponding to a profit of \$4000. Move the profit line as far from the origin as you can in order to determine which extreme point will provide the optimal solution. Compare your answer with the approach you used in part (b).
d. Which constraints are binding? Explain.
e. Suppose that the values of the objective function coefficients are \$4 for each All-Pro model produced and \$5 for each College model. Use the graphical solution procedure to determine the new optimal solution and the corresponding value of profit.

FIGURE 7.24 PORTION OF THE GRAPHICAL SOLUTION FOR EXERCISE 22



23. Embassy Motorcycles (EM) manufacturers two lightweight motorcycles designed for easy handling and safety. The EZ-Rider model has a new engine and a low profile that make it easy to balance. The Lady-Sport model is slightly larger, uses a more traditional engine, and is specifically designed to appeal to women riders. Embassy produces the engines for both models at its Des Moines, Iowa, plant. Each EZ-Rider engine requires 6 hours of manufacturing time and each Lady-Sport engine requires 3 hours of manufacturing time. The Des Moines plant has 2100 hours of engine manufacturing time available for the next production period. Embassy's motorcycle frame supplier can supply as many EZ-Rider frames as needed. However, the Lady-Sport frame is more complex and the supplier can only provide up to 280 Lady-Sport frames for the next production period. Final assembly and testing requires 2 hours for each EZ-Rider model and 2.5 hours for each Lady-Sport model. A maximum of 1000 hours of assembly and testing time are available for the next production period. The company's accounting department projects a profit contribution of $2400 for each EZ-Rider produced and $1800 for each Lady-Sport produced.
   a. Formulate a linear programming model that can be used to determine the number of units of each model that should be produced in order to maximize the total contribution to profit.
   b. Solve the problem graphically. What is the optimal solution?
   c. Which constraints are binding?

SELF test

24. Kelson Sporting Equipment, Inc., makes two different types of baseball gloves: a regular model and a catcher's model. The firm has 900 hours of production time available in its cutting and sewing department, 300 hours available in its finishing department, and 100 hours available in its packaging and shipping department. The production time requirements and the profit contribution per glove are given in the following table:

	Production Time (hours)			
Model	**Cutting and Sewing**	**Finishing**	**Packaging and Shipping**	**Profit/Glove**
Regular model	1	$^1/_2$	$^1/_8$	$5
Catcher's model	$^3/_2$	$^1/_3$	$^1/_4$	$8

Assuming that the company is interested in maximizing the total profit contribution, answer the following:
   a. What is the linear programming model for this problem?
   b. Find the optimal solution using the graphical solution procedure. How many gloves of each model should Kelson manufacture?
   c. What is the total profit contribution Kelson can earn with the given production quantities?
   d. How many hours of production time will be scheduled in each department?
   e. What is the slack time in each department?

25. George Johnson recently inherited a large sum of money; he wants to use a portion of this money to set up a trust fund for his two children. The trust fund has two investment options: (1) a bond fund and (2) a stock fund. The projected returns over the life of the investments are 6% for the bond fund and 10% for the stock fund. Whatever portion of the inheritance George finally decides to commit to the trust fund, he wants to invest at least 30% of that amount in the bond fund. In addition, he wants to select a mix that will enable him to obtain a total return of at least 7.5%.
   a. Formulate a linear programming model that can be used to determine the percentage that should be allocated to each of the possible investment alternatives.
   b. Solve the problem using the graphical solution procedure.

26. The Sea Wharf Restaurant would like to determine the best way to allocate a monthly advertising budget of $1000 between newspaper advertising and radio advertising. Management decided that at least 25% of the budget must be spent on each type of media, and that the amount of money spent on local newspaper advertising must be at least twice the amount spent on radio advertising. A marketing consultant developed an index that measures audience exposure per dollar of advertising on a scale from 0 to 100, with higher values implying greater audience exposure. If the value of the index for local newspaper advertising is 50 and the value of the index for spot radio advertising is 80, how should the restaurant allocate its advertising budget in order to maximize the value of total audience exposure?
    a. Formulate a linear programming model that can be used to determine how the restaurant should allocate its advertising budget in order to maximize the value of total audience exposure.
    b. Solve the problem using the graphical solution procedure.
27. Blair & Rosen, Inc. (B&R) is a brokerage firm that specializes in investment portfolios designed to meet the specific risk tolerances of its clients. A client who contacted B&R this past week has a maximum of $50,000 to invest. B&R's investment advisor decides to recommend a portfolio consisting of two investment funds: an Internet fund and a Blue Chip fund. The Internet fund has a projected annual return of 12%, while the Blue Chip fund has a projected annual return of 9%. The investment advisor requires that at most $35,000 of the client's funds should be invested in the Internet fund. B&R services include a risk rating for each investment alternative. The Internet fund, which is the more risky of the two investment alternatives, has a risk rating of 6 per thousand dollars invested. The Blue Chip fund has a risk rating of 4 per thousand dollars invested. For example, if $10,000 is invested in each of the two investment funds, B&R's risk rating for the portfolio would be $6(10) + 4(10) = 100$. Finally, B&R developed a questionnaire to measure each client's risk tolerance. Based on the responses, each client is classified as a conservative, moderate, or aggressive investor. Suppose that the questionnaire results classified the current client as a moderate investor. B&R recommends that a client who is a moderate investor limit his or her portfolio to a maximum risk rating of 240.
    a. What is the recommended investment portfolio for this client? What is the annual return for the portfolio?
    b. Suppose that a second client with $50,000 to invest has been classified as an aggressive investor. B&R recommends that the maximum portfolio risk rating for an aggressive investor is 320. What is the recommended investment portfolio for this aggressive investor? Discuss what happens to the portfolio under the aggressive investor strategy.
    c. Suppose that a third client with $50,000 to invest has been classified as a conservative investor. B&R recommends that the maximum portfolio risk rating for a conservative investor is 160. Develop the recommended investment portfolio for the conservative investor. Discuss the interpretation of the slack variable for the total investment fund constraint.
28. Tom's, Inc., produces various Mexican food products and sells them to Western Foods, a chain of grocery stores located in Texas and New Mexico. Tom's, Inc., makes two salsa products: Western Foods Salsa and Mexico City Salsa. Essentially, the two products have different blends of whole tomatoes, tomato sauce, and tomato paste. The Western Foods Salsa is a blend of 50% whole tomatoes, 30% tomato sauce, and 20% tomato paste. The Mexico City Salsa, which has a thicker and chunkier consistency, consists of 70% whole tomatoes, 10% tomato sauce, and 20% tomato paste. Each jar of salsa produced weighs 10 ounces. For the current production period, Tom's, Inc., can purchase up to 280 pounds of whole tomatoes, 130 pounds of tomato sauce, and 100 pounds of tomato paste; the price

per pound for these ingredients is \$0.96, \$0.64, and \$0.56, respectively. The cost of the spices and the other ingredients is approximately \$0.10 per jar. Tom's, Inc., buys empty glass jars for \$0.02 each, and labeling and filling costs are estimated to be \$0.03 for each jar of salsa produced. Tom's contract with Western Foods results in sales revenue of \$1.64 for each jar of Western Foods Salsa and \$1.93 for each jar of Mexico City Salsa.

a. Develop a linear programming model that will enable Tom's to determine the mix of salsa products that will maximize the total profit contribution.
b. Find the optimal solution.

29. AutoIgnite produces electronic ignition systems for automobiles at a plant in Cleveland, Ohio. Each ignition system is assembled from two components produced at AutoIgnite's plants in Buffalo, New York, and Dayton, Ohio. The Buffalo plant can produce 2000 units of component 1, 1000 units of component 2, or any combination of the two components each day. For instance, 60% of Buffalo's production time could be used to produce component 1 and 40% of Buffalo's production time could be used to produce component 2; in this case, the Buffalo plant would be able to produce $0.6(2000) = 1200$ units of component 1 each day and $0.4(1000) = 400$ units of component 2 each day. The Dayton plant can produce 600 units of component 1, 1400 units of component 2, or any combination of the two components each day. At the end of each day, the component production at Buffalo and Dayton is sent to Cleveland for assembly of the ignition systems on the following workday.
    a. Formulate a linear programming model that can be used to develop a daily production schedule for the Buffalo and Dayton plants that will maximize daily production of ignition systems at Cleveland.
    b. Find the optimal solution.

30. A financial advisor at Diehl Investments identified two companies that are likely candidates for a takeover in the near future. Eastern Cable is a leading manufacturer of flexible cable systems used in the construction industry, and ComSwitch is a new firm specializing in digital switching systems. Eastern Cable is currently trading for \$40 per share, and ComSwitch is currently trading for \$25 per share. If the takeovers occur, the financial advisor estimates that the price of Eastern Cable will go to \$55 per share and ComSwitch will go to \$43 per share. At this point in time, the financial advisor has identified ComSwitch as the higher-risk alternative. Assume that a client indicated a willingness to invest a maximum of \$50,000 in the two companies. The client wants to invest at least \$15,000 in Eastern Cable and at least \$10,000 in ComSwitch. Because of the higher risk associated with ComSwitch, the financial advisor has recommended that at most \$25,000 should be invested in ComSwitch.
    a. Formulate a linear programming model that can be used to determine the number of shares of Eastern Cable and the number of shares of ComSwitch that will meet the investment constraints and maximize the total return for the investment.
    b. Graph the feasible region.
    c. Determine the coordinates of each extreme point.
    d. Find the optimal solution.

SELF test

31. Consider the following linear program:

$$\begin{aligned} \text{Min} \quad & 3A + 4B \\ \text{s.t.} \quad & \\ & 1A + 3B \geq 6 \\ & 1A + 1B \geq 4 \\ & A, B \geq 0 \end{aligned}$$

Identify the feasible region and find the optimal solution using the graphical solution procedure. What is the value of the objective function?

32. Identify the three extreme-point solutions for the M&D Chemicals problem (see Section 7.5). Identify the value of the objective function and the values of the slack and surplus variables at each extreme point.

33. Consider the following linear programming problem:

$$\begin{array}{ll} \text{Min} & A + 2B \\ \text{s.t.} & \\ & A + 4B \leq 21 \\ & 2A + B \geq 7 \\ & 3A + 1.5B \leq 21 \\ & -2A + 6B \geq 0 \\ & A, B \geq 0 \end{array}$$

a. Find the optimal solution using the graphical solution procedure and the value of the objective function.
b. Determine the amount of slack or surplus for each constraint.
c. Suppose the objective function is changed to max $5A + 2B$. Find the optimal solution and the value of the objective function.

SELF test

34. Consider the following linear program:

$$\begin{array}{ll} \text{Min} & 2A + 2B \\ \text{s.t.} & \\ & 1A + 3B \leq 12 \\ & 3A + 1B \geq 13 \\ & 1A - 1B = 3 \\ & A, B \geq 0 \end{array}$$

a. Show the feasible region.
b. What are the extreme points of the feasible region?
c. Find the optimal solution using the graphical solution procedure.

SELF test

35. For the linear program

$$\begin{array}{ll} \text{Min} & 6A + 4B \\ \text{s.t.} & \\ & 2A + 1B \geq 12 \\ & 1A + 1B \geq 10 \\ & 1B \leq 4 \\ & A, B \geq 0 \end{array}$$

a. Write the problem in standard form.
b. Solve the problem using the graphical solution procedure.
c. What are the values of the slack and surplus variables?

36. As part of a quality improvement initiative, Consolidated Electronics employees complete a three-day training program on teaming and a two-day training program on problem solving. The manager of quality improvement has requested that at least 8 training programs on teaming and at least 10 training programs on problem solving be offered during the next six months. In addition, senior-level management has specified that at least 25 training programs must be offered during this period. Consolidated Electronics uses a consultant to teach the training programs. During the next quarter, the consultant has 84 days of

training time available. Each training program on teaming costs $10,000 and each training program on problem solving costs $8000.

a. Formulate a linear programming model that can be used to determine the number of training programs on teaming and the number of training programs on problem solving that should be offered in order to minimize total cost.
b. Graph the feasible region.
c. Determine the coordinates of each extreme point.
d. Solve for the minimum-cost solution.

37. The New England Cheese Company produces two cheese spreads by blending mild cheddar cheese with extra sharp cheddar cheese. The cheese spreads are packaged in 12-ounce containers, which are then sold to distributors throughout the Northeast. The Regular blend contains 80% mild cheddar and 20% extra sharp, and the Zesty blend contains 60% mild cheddar and 40% extra sharp. This year, a local dairy cooperative offered to provide up to 8100 pounds of mild cheddar cheese for $1.20 per pound and up to 3000 pounds of extra sharp cheddar cheese for $1.40 per pound. The cost to blend and package the cheese spreads, excluding the cost of the cheese, is $0.20 per container. If each container of Regular is sold for $1.95 and each container of Zesty is sold for $2.20, how many containers of Regular and Zesty should New England Cheese produce?

38. Applied Technology, Inc. (ATI) produces bicycle frames using two fiberglass materials that improve the strength-to-weight ratio of the frames. The cost of the standard-grade material is $7.50 per yard and the cost of the professional-grade material is $9.00 per yard. The standard- and professional-grade materials contain different amounts of fiberglass, carbon fiber, and Kevlar, as shown in the following table:

	Standard Grade	Professional Grade
Fiberglass	84%	58%
Carbon fiber	10%	30%
Kevlar	6%	12%

ATI signed a contract with a bicycle manufacturer to produce a new frame with a carbon fiber content of at least 20% and a Kevlar content of not greater than 10%. To meet the required weight specification, a total of 30 yards of material must be used for each frame.

a. Formulate a linear program to determine the number of yards of each grade of fiberglass material that ATI should use in each frame in order to minimize total cost. Define the decision variables and indicate the purpose of each constraint.
b. Use the graphical solution procedure to determine the feasible region. What are the coordinates of the extreme points?
c. Compute the total cost at each extreme point. What is the optimal solution?
d. The distributor of the fiberglass material is currently overstocked with the professional-grade material. To reduce inventory, the distributor offered ATI the opportunity to purchase the professional-grade material for $8 per yard. Will the optimal solution change?
e. Suppose that the distributor further lowers the price of the professional-grade material to $7.40 per yard. Will the optimal solution change? What effect would an even lower price for the professional-grade material have on the optimal solution? Explain.

39. Innis Investments manages funds for a number of companies and wealthy clients. The investment strategy is tailored to each client's needs. For a new client, Innis has been authorized to invest up to $1.2 million in two investment funds: a stock fund and a money market

fund. Each unit of the stock fund costs \$50 and provides an annual rate of return of 10%; each unit of the money market fund costs \$100 and provides an annual rate of return of 4%.

The client wants to minimize risk subject to the requirement that the annual income from the investment be at least \$60,000. According to Innis's risk measurement system, each unit invested in the stock fund has a risk index of 8, and each unit invested in the money market fund has a risk index of 3; the higher risk index associated with the stock fund simply indicates that it is the riskier investment. Innis's client also specifies that at least \$300,000 be invested in the money market fund.

a. Determine how many units of each fund Innis should purchase for the client to minimize the total risk index for the portfolio.
b. How much annual income will this investment strategy generate?
c. Suppose the client desires to maximize annual return. How should the funds be invested?

40. Eastern Chemicals produces two types of lubricating fluids used in industrial manufacturing. Both products cost Eastern Chemicals \$1 per gallon to produce. Based on an analysis of current inventory levels and outstanding orders for the next month, Eastern Chemicals' management specified that at least 30 gallons of product 1 and at least 20 gallons of product 2 must be produced during the next two weeks. Management also stated that an existing inventory of highly perishable raw material required in the production of both fluids must be used within the next two weeks. The current inventory of the perishable raw material is 80 pounds. Although more of this raw material can be ordered if necessary, any of the current inventory that is not used within the next two weeks will spoil—hence, the management requirement that at least 80 pounds be used in the next two weeks. Furthermore, it is known that product 1 requires 1 pound of this perishable raw material per gallon and product 2 requires 2 pounds of the raw material per gallon. Because Eastern Chemicals' objective is to keep its production costs at the minimum possible level, the firm's management is looking for a minimum-cost production plan that uses all the 80 pounds of perishable raw material and provides at least 30 gallons of product 1 and at least 20 gallons of product 2. What is the minimum-cost solution?

41. Southern Oil Company produces two grades of gasoline: regular and premium. The profit contributions are \$0.30 per gallon for regular gasoline and \$0.50 per gallon for premium gasoline. Each gallon of regular gasoline contains 0.3 gallons of grade A crude oil and each gallon of premium gasoline contains 0.6 gallons of grade A crude oil. For the next production period, Southern has 18,000 gallons of grade A crude oil available. The refinery used to produce the gasolines has a production capacity of 50,000 gallons for the next production period. Southern Oil's distributors have indicated that demand for the premium gasoline for the next production period will be at most 20,000 gallons.

a. Formulate a linear programming model that can be used to determine the number of gallons of regular gasoline and the number of gallons of premium gasoline that should be produced in order to maximize total profit contribution.
b. What is the optimal solution?
c. What are the values and interpretations of the slack variables?
d. What are the binding constraints?

SELF test

42. Does the following linear program involve infeasibility, unbounded, and/or alternative optimal solutions? Explain.

$$
\begin{aligned}
\text{Max} \quad & 4A + 8B \\
\text{s.t.} \quad & \\
& 2A + 2B \leq 10 \\
& -1A + 1B \geq 8 \\
& A, B \geq 0
\end{aligned}
$$

SELF test

43. Does the following linear program involve infeasibility, unbounded, and/or alternative optimal solutions? Explain.

$$\begin{aligned} \text{Max} \quad & 1A + 1B \\ \text{s.t.} \quad & \\ & 8A + 6B \geq 24 \\ & 2B \geq 4 \\ & A, B \geq 0 \end{aligned}$$

44. Consider the following linear program:

$$\begin{aligned} \text{Max} \quad & 1A + 1B \\ \text{s.t.} \quad & \\ & 5A + 3B \leq 15 \\ & 3A + 5B \leq 15 \\ & A, B \geq 0 \end{aligned}$$

a. What is the optimal solution for this problem?
b. Suppose that the objective function is changed to $1A + 2B$. Find the new optimal solution.

45. Consider the following linear program:

$$\begin{aligned} \text{Max} \quad & 1A - 2B \\ \text{s.t.} \quad & \\ & -4A + 3B \leq 3 \\ & 1A - 1B \leq 3 \\ & A, B \geq 0 \end{aligned}$$

a. Graph the feasible region for the problem.
b. Is the feasible region unbounded? Explain.
c. Find the optimal solution.
d. Does an unbounded feasible region imply that the optimal solution to the linear program will be unbounded?

46. The manager of a small independent grocery store is trying to determine the best use of her shelf space for soft drinks. The store carries national and generic brands and currently has 200 square feet of shelf space available. The manager wants to allocate at least 60% of the space to the national brands and, regardless of the profitability, allocate at least 10% of the space to the generic brands. How many square feet of space should the manager allocate to the national brands and the generic brands under the following circumstances?
a. The national brands are more profitable than the generic brands.
b. Both brands are equally profitable.
c. The generic brand is more profitable than the national brand.

47. Discuss what happens to the M&D Chemicals problem (see Section 7.5) if the cost per gallon for product A is increased to $3.00 per gallon. What would you recommend? Explain.

48. For the M&D Chemicals problem in Section 7.5, discuss the effect of management's requiring total production of 500 gallons for the two products. List two or three actions M&D should consider to correct the situation you encounter.

49. PharmaPlus operates a chain of 30 pharmacies. The pharmacies are staffed by licensed pharmacists and pharmacy technicians. The company currently employs 85 full-time-equivalent

pharmacists (combination of full time and part time) and 175 full-time-equivalent technicians. Each spring management reviews current staffing levels and makes hiring plans for the year. A recent forecast of the prescription load for the next year shows that at least 250 full-time-equivalent employees (pharmacists and technicians) will be required to staff the pharmacies. The personnel department expects 10 pharmacists and 30 technicians to leave over the next year. To accommodate the expected attrition and prepare for future growth, management states that at least 15 new pharmacists must be hired. In addition, PharmaPlus's new service quality guidelines specify no more than two technicians per licensed pharmacist. The average salary for licensed pharmacists is $40 per hour and the average salary for technicians is $10 per hour.

a. Determine a minimum-cost staffing plan for PharmaPlus. How many pharmacists and technicians are needed?
b. Given current staffing levels and expected attrition, how many new hires (if any) must be made to reach the level recommended in part (a)? What will be the impact on the payroll?

50. Expedition Outfitters manufactures a variety of specialty clothing for hiking, skiing, and mountain climbing. The company has decided to begin production on two new parkas designed for use in extremely cold weather: the Mount Everest Parka and the Rocky Mountain Parka. Expedition's manufacturing plant has 120 hours of cutting time and 120 hours of sewing time available for producing these two parkas. Each Mount Everest Parka requires 30 minutes of cutting time and 45 minutes of sewing time, and each Rocky Mountain Parka requires 20 minutes of cutting time and 15 minutes of sewing time. The labor and material cost is $150 for each Mount Everest Parka and $50 for each Rocky Mountain Parka, and the retail prices through the firm's mail order catalog are $250 for the Mount Everest Parka and $200 for the Rocky Mountain Parka. Because management believes that the Mount Everest Parka is a unique coat that will enhance the image of the firm, management specified that at least 20% of the total production must consist of this model. Assuming that Expedition Outfitters can sell as many coats of each type as it can produce, how many units of each model should it manufacture to maximize the total profit contribution?

51. English Motors, Ltd. (EML), developed a new all-wheel-drive sports utility vehicle. As part of the marketing campaign, EML produced a digitally recorded sales presentation to send to both owners of current EML four-wheel-drive vehicles as well as to owners of four-wheel-drive sports utility vehicles offered by competitors; EML refers to these two target markets as the current customer market and the new customer market. Individuals who receive the new promotion will also receive a coupon for a test drive of the new EML model for one weekend. A key factor in the success of the new promotion is the response rate, the percentage of individuals who receive the new promotion and test drive the new model. EML estimates that the response rate for the current customer market is 25% and the response rate for the new customer market is 20%. For the customers who test drive the new model, the sales rate is the percentage of individuals who make a purchase. Marketing research studies indicate that the sales rate is 12% for the current customer market and 20% for the new customer market. The cost for each promotion, excluding the test drive costs, is $4 for each promotion sent to the current customer market and $6 for each promotion sent to the new customer market. Management also specified that a minimum of 30,000 current customers should test drive the new model and a minimum of 10,000 new customers should test drive the new model. In addition, the number of current customers who test drive the new vehicle must be at least twice the number of new customers who test drive the new vehicle. If the marketing budget, excluding test drive costs, is $1.2 million, how many promotions should be sent to each group of customers in order to maximize total sales?

52. Creative Sports Design (CSD) manufactures a standard-size racket and an oversize racket. The firm's rackets are extremely light due to the use of a magnesium-graphite alloy that was invented by the firm's founder. Each standard-size racket uses 0.125 kilograms of the alloy and each oversize racket uses 0.4 kilograms; over the next two-week production period, only 80 kilograms of the alloy are available. Each standard-size racket uses 10 minutes of manufacturing time and each oversize racket uses 12 minutes. The profit contributions are \$10 for each standard-size racket and \$15 for each oversize racket, and 40 hours of manufacturing time are available each week. Management specified that at least 20% of the total production must be the standard-size racket. How many rackets of each type should CSD manufacture over the next two weeks to maximize the total profit contribution? Assume that because of the unique nature of its products, CSD can sell as many rackets as it can produce.

53. Management of High Tech Services (HTS) would like to develop a model that will help allocate its technicians' time between service calls to regular contract customers and new customers. A maximum of 80 hours of technician time is available over the two-week planning period. To satisfy cash flow requirements, at least \$800 in revenue (per technician) must be generated during the two-week period. Technician time for regular customers generates \$25 per hour. However, technician time for new customers only generates an average of \$8 per hour because in many cases a new customer contact does not provide billable services. To ensure that new customer contacts are being maintained, the technician time spent on new customer contacts must be at least 60% of the time spent on regular customer contacts. Given these revenue and policy requirements, HTS would like to determine how to allocate technician time between regular customers and new customers so that the total number of customers contacted during the two-week period will be maximized. Technicians require an average of 50 minutes for each regular customer contact and 1 hour for each new customer contact.
    a. Develop a linear programming model that will enable HTS to allocate technician time between regular customers and new customers.
    b. Find the optimal solution.

54. Jackson Hole Manufacturing is a small manufacturer of plastic products used in the automotive and computer industries. One of its major contracts is with a large computer company and involves the production of plastic printer cases for the computer company's portable printers. The printer cases are produced on two injection molding machines. The M-100 machine has a production capacity of 25 printer cases per hour, and the M-200 machine has a production capacity of 40 cases per hour. Both machines use the same chemical material to produce the printer cases; the M-100 uses 40 pounds of the raw material per hour and the M-200 uses 50 pounds per hour. The computer company asked Jackson Hole to produce as many of the cases during the upcoming week as possible; it will pay \$18 for each case Jackson Hole can deliver. However, next week is a regularly scheduled vacation period for most of Jackson Hole's production employees; during this time, annual maintenance is performed for all equipment in the plant. Because of the downtime for maintenance, the M-100 will be available for no more than 15 hours, and the M-200 will be available for no more than 10 hours. However, because of the high set-up cost involved with both machines, management requires that, if production is scheduled on either machine, the machine must be operated for at least 5 hours. The supplier of the chemical material used in the production process informed Jackson Hole that a maximum of 1000 pounds of the chemical material will be available for next week's production; the cost for this raw material is \$6 per pound. In addition to the raw material cost, Jackson Hole estimates that the hourly costs of operating the M-100 and the M-200 are \$50 and \$75, respectively.

a. Formulate a linear programming model that can be used to maximize the contribution to profit.
b. Find the optimal solution.

Case Problem 1 Workload Balancing

Digital Imaging (DI) produces photo printers for both the professional and consumer markets. The DI consumer division recently introduced two photo printers that provide color prints rivaling those produced by a professional processing lab. The DI-910 model can produce a 4″ × 6″ borderless print in approximately 37 seconds. The more sophisticated and faster DI-950 can even produce a 13″ × 19″ borderless print. Financial projections show profit contributions of $42 for each DI-910 and $87 for each DI-950.

The printers are assembled, tested, and packaged at DI's plant located in New Bern, North Carolina. This plant is highly automated and uses two manufacturing lines to produce the printers. Line 1 performs the assembly operation with times of 3 minutes per DI-910 printer and 6 minutes per DI-950 printer. Line 2 performs both the testing and packaging operations. Times are 4 minutes per DI-910 printer and 2 minutes per DI-950 printer. The shorter time for the DI-950 printer is a result of its faster print speed. Both manufacturing lines are in operation one 8-hour shift per day.

Managerial Report

Perform an analysis for Digital Imaging in order to determine how many units of each printer to produce. Prepare a report to DI's president presenting your findings and recommendations. Include (but do not limit your discussion to) a consideration of the following:

1. The recommended number of units of each printer to produce to maximize the total contribution to profit for an 8-hour shift. What reasons might management have for not implementing your recommendation?
2. Suppose that management also states that the number of DI-910 printers produced must be at least as great as the number of DI-950 units produced. Assuming that the objective is to maximize the total contribution to profit for an 8-hour shift, how many units of each printer should be produced?
3. Does the solution you developed in part (2) balance the total time spent on line 1 and the total time spent on line 2? Why might this balance or lack of it be a concern to management?
4. Management requested an expansion of the model in part (2) that would provide a better balance between the total time on line 1 and the total time on line 2. Management wants to limit the difference between the total time on line 1 and the total time on line 2 to 30 minutes or less. If the objective is still to maximize the total contribution to profit, how many units of each printer should be produced? What effect does this workload balancing have on total profit in part (2)?
5. Suppose that in part (1) management specified the objective of maximizing the total number of printers produced each shift rather than total profit contribution. With this objective, how many units of each printer should be produced per shift? What effect does this objective have on total profit and workload balancing?

For each solution that you develop, include a copy of your linear programming model and graphical solution in the appendix to your report.

Case Problem 2 Production Strategy

Better Fitness, Inc. (BFI) manufactures exercise equipment at its plant in Freeport, Long Island. It recently designed two universal weight machines for the home exercise market. Both machines use BFI-patented technology that provides the user with an extremely wide range of motion capability for each type of exercise performed. Until now, such capabilities have been available only on expensive weight machines used primarily by physical therapists.

At a recent trade show, demonstrations of the machines resulted in significant dealer interest. In fact, the number of orders that BFI received at the trade show far exceeded its manufacturing capabilities for the current production period. As a result, management decided to begin production of the two machines. The two machines, which BFI named the BodyPlus 100 and the BodyPlus 200, require different amounts of resources to produce.

The BodyPlus 100 consists of a frame unit, a press station, and a pec-dec station. Each frame produced uses 4 hours of machining and welding time and 2 hours of painting and finishing time. Each press station requires 2 hours of machining and welding time and 1 hour of painting and finishing time, and each pec-dec station uses 2 hours of machining and welding time and 2 hours of painting and finishing time. In addition, 2 hours are spent assembling, testing, and packaging each BodyPlus 100. The raw material costs are $450 for each frame, $300 for each press station, and $250 for each pec-dec station; packaging costs are estimated to be $50 per unit.

The BodyPlus 200 consists of a frame unit, a press station, a pec-dec station, and a leg-press station. Each frame produced uses 5 hours of machining and welding time and 4 hours of painting and finishing time. Each press station requires 3 hours machining and welding time and 2 hours of painting and finishing time, each pec-dec station uses 2 hours of machining and welding time and 2 hours of painting and finishing time, and each leg-press station requires 2 hours of machining and welding time and 2 hours of painting and finishing time. In addition, 2 hours are spent assembling, testing, and packaging each BodyPlus 200. The raw material costs are $650 for each frame, $400 for each press station, $250 for each pec-dec station, and $200 for each leg-press station; packaging costs are estimated to be $75 per unit.

For the next production period, management estimates that 600 hours of machining and welding time; 450 hours of painting and finishing time; and 140 hours of assembly, testing, and packaging time will be available. Current labor costs are $20 per hour for machining and welding time; $15 per hour for painting and finishing time; and $12 per hour for assembly, testing, and packaging time. The market in which the two machines must compete suggests a retail price of $2400 for the BodyPlus 100 and $3500 for the BodyPlus 200, although some flexibility may be available to BFI because of the unique capabilities of the new machines. Authorized BFI dealers can purchase machines for 70% of the suggested retail price.

BFI's president believes that the unique capabilities of the BodyPlus 200 can help position BFI as one of the leaders in high-end exercise equipment. Consequently, she states that the number of units of the BodyPlus 200 produced must be at least 25% of the total production.

Managerial Report

Analyze the production problem at Better Fitness, Inc., and prepare a report for BFI's president presenting your findings and recommendations. Include (but do not limit your discussion to) a consideration of the following items:

1. The recommended number of BodyPlus 100 and BodyPlus 200 machines to produce

2. The effect on profits of the requirement that the number of units of the BodyPlus 200 produced must be at least 25% of the total production
3. Where efforts should be expended in order to increase contribution to profits

Include a copy of your linear programming model and graphical solution in an appendix to your report.

Case Problem 3 Hart Venture Capital

Hart Venture Capital (HVC) specializes in providing venture capital for software development and Internet applications. Currently HVC has two investment opportunities: (1) Security Systems, a firm that needs additional capital to develop an Internet security software package, and (2) Market Analysis, a market research company that needs additional capital to develop a software package for conducting customer satisfaction surveys. In exchange for Security Systems stock, the firm asked HVC to provide \$600,000 in year 1, \$600,000 in year 2, and \$250,000 in year 3 over the coming three-year period. In exchange for Market Analysis stock, the firm asked HVC to provide \$500,000 in year 1, \$350,000 in year 2, and \$400,000 in year 3 over the same three-year period. HVC believes that both investment opportunities are worth pursuing. However, because of other investments, HVC is willing to commit at most \$800,000 for both projects in the first year, at most \$700,000 in the second year, and \$500,000 in the third year.

HVC's financial analysis team reviewed both projects and recommended that the company's objective should be to maximize the net present value of the total investment in Security Systems and Market Analysis. The net present value takes into account the estimated value of the stock at the end of the three-year period as well as the capital outflows that are necessary during each of the three years. Using an 8% rate of return, HVC's financial analysis team estimates that 100% funding of the Security Systems project has a net present value of \$1,800,000, and 100% funding of the Market Analysis project has a net present value of \$1,600,000.

HVC has the option to fund any percentage of the Security Systems and Market Analysis projects. For example, if HVC decides to fund 40% of the Security Systems project, investments of 0.40(\$600,000) = \$240,000 would be required in year 1, 0.40(\$600,000) = \$240,000 would be required in year 2, and 0.40(\$250,000) = \$100,000 would be required in year 3. In this case, the net present value of the Security Systems project would be 0.40(\$1,800,000) = \$720,000. The investment amounts and the net present value for partial funding of the Market Analysis project would be computed in the same manner.

Managerial Report

Perform an analysis of HVC's investment problem and prepare a report that presents your findings and recommendations. Be sure to include information on the following:

1. The recommended percentage of each project that HVC should fund and the net present value of the total investment
2. A capital allocation plan for Security Systems and Market Analysis for the coming three-year period and the total HVC investment each year
3. The effect, if any, on the recommended percentage of each project that HVC should fund if HVC is willing to commit an additional \$100,000 during the first year
4. A capital allocation plan if an additional \$100,000 is made available
5. Your recommendation as to whether HVC should commit the additional \$100,000 in the first year

Provide model details and relevant computer output in a report appendix.

Appendix 7.1 Solving Linear Programs with Excel 2010

In this appendix we will use an Excel worksheet to solve the RMC linear programming problem. We will enter the problem data for the RMC problem in the top part of the worksheet and develop the linear programming model in the bottom part of the worksheet. Note that Appendix A contains much more detail on how to formulate models in Excel.

Formulation

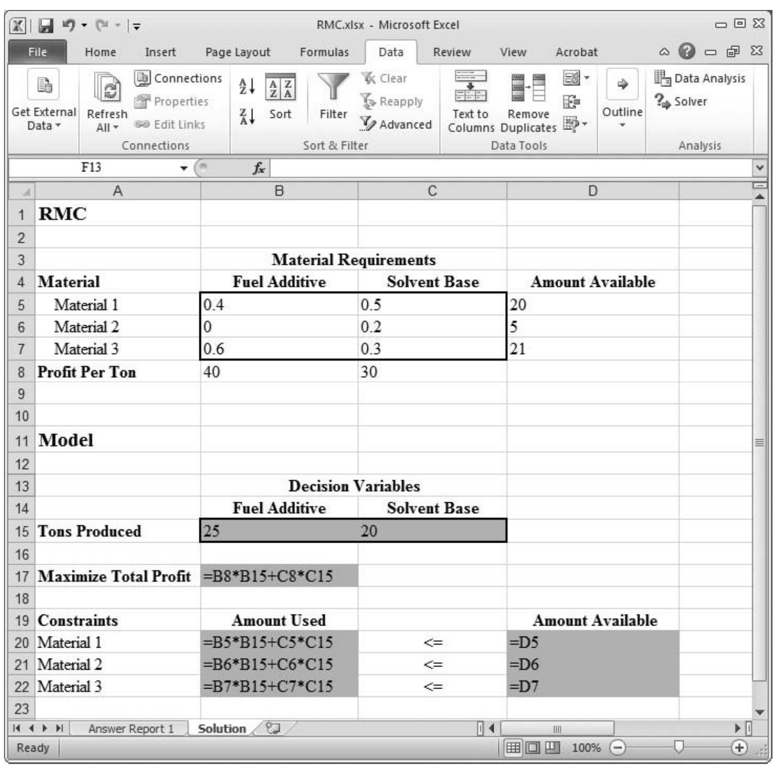
Whenever we formulate a worksheet model of a linear program, we perform the following steps:

Step 1. Enter the problem data in the top part of the worksheet
Step 2. Specify cell locations for the decision variables
Step 3. Select a cell and enter a formula for computing the value of the objective function
Step 4. Select a cell and enter a formula for computing the left-hand side of each constraint
Step 5. Select a cell and enter a formula for computing the right-hand side of each constraint

The formula worksheet that we developed for the RMC problem using these five steps is shown in Figure 7.25. Let us review each of the preceding steps as they apply to the RMC problem.

FIGURE 7.25 EXCEL FORMULA WORKSHEET FOR THE RMC PROBLEM

WEB file
RMC



	A	B	C	D
1	RMC			
2				
3		Material Requirements		
4	Material	Fuel Additive	Solvent Base	Amount Available
5	Material 1	0.4	0.5	20
6	Material 2	0	0.2	5
7	Material 3	0.6	0.3	21
8	Profit Per Ton	40	30	
9				
10				
11	Model			
12				
13		Decision Variables		
14		Fuel Additive	Solvent Base	
15	Tons Produced	25	20	
16				
17	Maximize Total Profit	=B8*B15+C8*C15		
18				
19	Constraints	Amount Used		Amount Available
20	Material 1	=B5*B15+C5*C15	<=	=D5
21	Material 2	=B6*B15+C6*C15	<=	=D6
22	Material 3	=B7*B15+C7*C15	<=	=D7
23				

Step 1. Enter the problem data in the top part of the worksheet
Cells B5 to C7 show the material requirements per ton of each product.
Cells B8 and C8 show the profit contribution per ton for the two products.
Cells D5 to D7 show the maximum amounts available for each of the three materials.

Step 2. Specify cell locations for the decision variables
Cell B15 will contain the number of tons of fuel additive produced, and Cell C15 will contain the number of tons of solvent base produced.

Step 3. Select a cell and enter a formula for computing the value of the objective function
Cell B17: =B8*B15+C8*C15

Step 4. Select a cell and enter a formula for computing the left-hand side of each constraint. With three constraints, we have
Cell B20: =B5*B15+C5*C15
Cell B21: =C6*C15
Cell B22: =B7*B15+C7*C15

Step 5. Select a cell and enter a formula for computing the right-hand side of each constraint. With three constraints, we have
Cell D20: =D5
Cell D21: =D6
Cell D22: =D7

Note that descriptive labels make the model section of the worksheet easier to read and understand. For example, we added "Fuel Additive," "Solvent Base," and "Tons Produced" in rows 14 and 15 so that the values of the decision variables appearing in Cells B15 and C15 can be easily interpreted. In addition, we entered "Maximize Total Profit" in Cell A17 to indicate that the value of the objective function appearing in Cell B17 is the maximum profit contribution. In the constraint section of the worksheet we added the constraint names as well as the "<=" symbols to show the relationship that exists between the left-hand side and the right-hand side of each constraint. Although these descriptive labels are not necessary to use Excel Solver to find a solution to the RMC problem, the labels make it easier for the user to understand and interpret the optimal solution.

Excel Solution

Excel 2010 contains an updated Solver capability that incorporates much of what was previously known as Premium Solver. Excel Solver, developed by Frontline Systems, can be used to solve all of the linear programming problems presented in this text.

The following steps describe how Excel Solver can be used to obtain the optimal solution to the RMC problem:

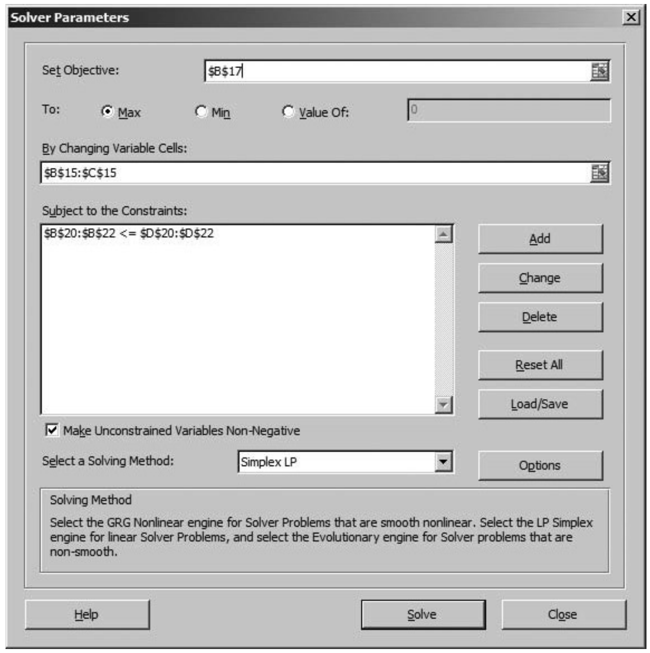
Step 1. Select the **Data** tab from the **Ribbon**

Step 2. Select **Solver** from the **Analysis Group** (see Figure 7.25, where the Analysis Group and Data tab are displayed in the Ribbon)

Step 3. When the **Solver Parameters** dialog box appears (see Figure 7.26):
Enter B17 into the **Set Objective** box
Select the **To: Max** option
Enter B15:C15 into the **By Changing Variable Cells** box
Select **Add**

Step 4. When the **Add Constraint** dialog box appears:
Enter B20:B22 in the **Cell Reference** box

FIGURE 7.26 EXCEL SOLVER PARAMETERS DIALOG BOX FOR THE RMC PROBLEM



Select <=
Enter D20:D22 in the **Constraint** box
Click **OK**

Step 5. When the **Solver Parameters** dialog box reappears:
Click the checkbox for **Make Unconstrained Variables Non-negative**

Step 6. Select the **Select a Solving Method** drop-down button
Select **Simplex LP***

Step 7. Click **Solve**

Step 8. When the **Solver Results** dialog box appears:
Select **Keep Solver Solution**
Click **OK**

*The Excel Answer Report that is similar to Figure 7.15 is generated from the **Solver Results** dialog box. This is created by clicking on **Answer** in the **Reports** group before clicking **OK** in step 7. We will discuss the Sensitivity Report in Chapter 8.*

Figure 7.26 shows the completed Excel Solver Parameters dialog box, and Figure 7.27 shows the optimal solution in the worksheet. The optimal solution of 25 tons of fuel additive and 20 tons of solvent base is the same as we obtained using the graphical solution procedure. Solver also has an option to provide sensitivity analysis information. We discuss sensitivity analysis in Chapter 8.

In step 5 we selected the **Make Unconstrained Variables Non-negative** option in the **Solver Parameters** dialog box to avoid having to enter nonnegativity constraints for the decision variables. In general, whenever we want to solve a linear programming model in

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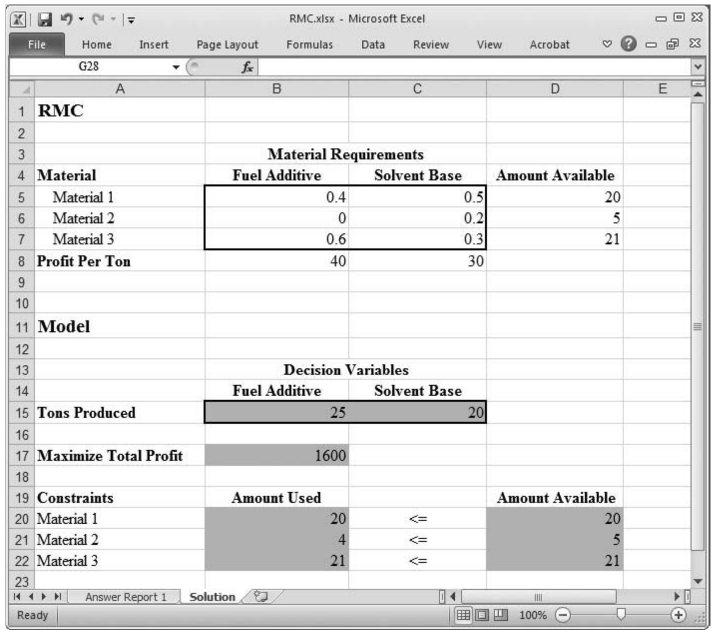
*Because we know this is a linear program, the Simplex LP algorithm is the most efficient solution method.

FIGURE 7.27 EXCEL SOLUTION FOR THE RMC PROBLEM



which the decision variables are all restricted to be nonnegative, we will select this option. In addition, in step 4 we entered all three less-than-or-equal-to constraints simultaneously by entering B20:B22 into the **Cell Reference** box, selecting $<=$, and entering D20:D22 into the **Constraint** box. Alternatively, we could have entered the three constraints one at a time.

The Solver Add-In should be found under the **Data** tab on the Excel Ribbon. If it does not appear here, you will have to add it by following the steps shown below.

Step 1. Select the **File** tab from the Ribbon
Step 2. Select **Options** from the **File** menu
Step 3. Choose **Add-Ins** from the Excel Options screen
Step 4. Click on **Go** next to **Manage: Excel Add-ins**
Step 5. Click on the **Solver Add-in** checkbox

Appendix 7.2 Solving Linear Programs with LINGO

In this appendix we describe how to use LINGO to solve the RMC problem. When you start LINGO, two windows are immediately displayed. The outer, or mainframe, window contains all the command menus and the command toolbar. The smaller window is the model window; this window is used to enter and edit the linear programming model you want to solve.

As with any model, it is good to document your LINGO model with comments. A comment in a LINGO model begins with an exclamation point and ends with a semicolon. If desired, a comment can span multiple lines.

The first item we enter is a comment describing the objective function. Recall that the objective function for the RMC problem is to maximize profit. Hence we enter the following comment:

```
! MAXIMIZE PROFIT;
```

For the latest information on LINGO software see http://www.lindo.com.

Next we press the Enter key and then type the objective function. The objective function for the RMC problem is Max $40F + 30S$. Thus, in the second line of the LINGO model window, we enter the following expression:

```
MAX = 40*F + 30*S;
```

Note that in LINGO the symbol * is used to denote multiplication and that the objective function, like a comment, ends with a semicolon. In general, each mathematical expression (objective function and constraints) in LINGO is terminated with a semicolon.

Next, we press the Enter key to move to a new line. The first constraint in the RMC problem is $0.4F + 0.5S \leq 20$, for material 1. Thus, in the third and fourth lines of the LINGO model window, we enter the following expressions:

```
!MATERIAL 1 CONSTRAINT;
0.4*F + 0.5*S <= 20;
```

Note that LINGO interprets the <= symbol as ≤. Alternatively, we could enter < instead of <=. As was the case when entering the objective function, a semicolon is required at the end of the first constraint. Pressing the Enter key moves us to a new line, and we continue the process by entering the remaining comments and constraints as shown here:

```
!MATERIAL 2 CONSTRAINT;
0.2*S <= 5;
!MATERIAL 3 CONSTRAINT;
0.6*F + 0.3*S <= 21;
```

The model window will now appear as follows:

```
!MAXIMIZE PROFIT;
MAX = 40*F + 30*S;
!MATERIAL 1 CONSTRAINT;
0.4*F + 0.5*S <= 20;
!MATERIAL 2 CONSTRAINT;
0.2*S <= 5;
!MATERIAL 3 CONSTRAINT;
0.6*F + 0.3*S <= 21;
```

If you make an error in entering the model, you can correct it at any time by simply positioning the cursor where you made the error and entering the necessary correction.

To solve the model, select the **Solve** command from the **LINGO** menu or press the **Solve** button on the toolbar at the top of the mainframe window. LINGO will begin the solution process by determining whether the model conforms to all syntax requirements. If the LINGO model doesn't pass these tests, you will be informed by an error message. If LINGO does not find any errors in the model input, it will begin to solve the model. As part of the solution process, LINGO displays a **Solver Status** window that allows you to monitor the progress of the solver. LINGO displays the solution in a new window titled "Solution Report." The output that appears in the **Solution Report** window for the RMC problem is shown in Figure 7.28.

FIGURE 7.28 SOLUTION TO THE RMC PROBLEM USING LINGO

```
Global optimal solution found.
Objective value:                                  1600.000
Total solver iterations:                                 2

                     Variable           Value        Reduced Cost
                            F        25.00000            0.000000
                            S        20.00000            0.000000

                          Row    Slack or Surplus      Dual Price
                            1        1600.000            1.000000
                            2        0.000000            33.33333
                            3        1.000000            0.000000
                            4        0.000000            44.44444
```


The first part of the output shown in Figure 7.28 indicates that an optimal solution has been found and that the value of the objective function is 1600. We see that the optimal solution is $F = 25$ and $S = 20$, and that the slack variables for the three constraints (rows 2–4) are 0, 1, and 0. We will discuss the use of the information in the Reduced Cost column and the Dual Price column in Chapter 8.